which is indeed completely covariant. We have used Eq. (7.89), relating H and $\mathcal{H}$, and set $R_{+-}=R$.

As in the bosonic string, this expression simplifies considerably if we represent superconformal classes by constant-curvature geometries and restrict $\mathcal{H}$ to elements in $\mathrm{Ker}\mathcal{P}_1^\dagger$. Recalling that the super Weil-Petersson metric on supermoduli is given by Eq. (3.44) with constant R_{+-}, we obtain at once that

$$\delta_{\mathcal{H}}\delta_{\overline{\mathcal{H}}}\ln\frac{\mathrm{sdet}'\Box_n^{(-)}}{\mathrm{sdet}\langle\Phi_j\,|\,\Phi_k\rangle\langle\Psi_\alpha\,|\,\Psi_\beta\rangle} = -7\frac{4n-1}{4\pi}(-)^{2n}\|\mathcal{H}\|^2_{\mathrm{sWP}}\,. \quad (7.110)$$

The above holomorphic coordinates for supermoduli space, second variations of superdeterminants, and holomorphic splitting of the superstring measure appear in D'Hoker and Phong (1987a). The approach taken there is in the supergeometry formalism. A different argument in Wess-Zumino gauge, also leading to holomorphic splitting, was provided later by Sonoda (1987d) and Bershadsky (1988). That Howe's solutions (3.32) provide holomorphic coordinates for supermoduli space in the sense of Eqs. (7.77) and (7.78), when the zweibein depends holomorphically on moduli, was verified by Nelson (1987b). Independent approaches to ratios of superdeterminants and superholomorphic splitting are discussed by Baranov and Schwarz (1987).

G. Global issues for the superstring

In this section we shall tie together the various properties of string amplitudes uncovered so far and propose a solution for a number of contradictions and ambiguities that have seemed to affect superstrings.

1. Chiral and superholomorphic splitting

We have discussed a number of different approaches to the splitting of string amplitudes as a function of left and right chirality degrees of freedom in Secs. III.K and III.M–III.O, or as a function of holomorphic and antiholomorphic dependence on moduli space in Secs. VII.A and VII.C–VII.E, or finally as a function of superholomorphic and antisuperholomorphic dependence on supermoduli space in Sec. VII.G—the previous section. The question thus arises whether all such approaches are the same or, if they are different, which one is correct. To discuss this, we shall make a finer distinction.

(a) Holomorphic splitting over moduli space

$$\text{amplitude}=\int_{\mathcal{M}_h} dm_k d\overline{m}_k\,|\,\mathcal{F}(m_k)\,|^2\,.$$

(b) Holomorphic splitting on moduli space at fixed internal momenta

$$\text{amplitude}=\int_{\mathfrak{F}} dp_I^\mu \int_{\mathcal{M}_h} dm_k d\overline{m}_k\,|\,\mathcal{F}(m_k,p_I^\mu)\,|^2\,.$$

(c) Chiral splitting at fixed internal momenta (and for odd-spin structure at fixed Dirac zero modes $\psi_\pm^{0\mu}$)

$$\text{amplitude}=\int_{\mathfrak{F}} dp_I^\mu \int_{s\mathcal{M}_h} d^2m_K\,|\,\mathcal{F}(\Omega,\chi_{\bar z}{}^+;p_I^\mu)\,|^2 \quad \text{even}\,,$$

$$\text{amplitude}=\int_{\mathfrak{F}} dp_I^\mu \int d\psi_\pm^{0\mu} \int_{s\mathcal{M}_h} d^2m_K\,|\,\mathcal{F}(\Omega,\chi_{\bar z}{}^+;\psi_\pm^{0\mu};p_I^\mu)\,|^2 \quad \text{odd}\,.$$

(d) Superholomorphic splitting on supermoduli space

$$\text{amplitude}=\int_{s\mathcal{M}_h} d^2m_K\,|\,\mathcal{F}(m_K)\,|^2\,.$$

(e) Superholomorphic splitting on supermoduli space at fixed internal momenta (and fixed Dirac zero modes for odd-spin structure)

$$\text{amplitude}=\int_{\mathfrak{F}} dp_I^\mu \left[\int d\psi_\mp^{0\mu}\right] \times \int_{s\mathcal{M}_h} d^2m_K\,|\,\mathcal{F}(m_K(;\psi_\pm^0);p_I^\mu)\,|^2\,.$$

Of course it is understood that the absolute value square is taken for the nonchiral theory.

We now discuss the validity and interrelation of the various possibilities.

(a) Holds for the partition function of the bosonic string only, provided the measure is defined to include the factor $\det(\mathrm{Im}\Omega)$. It does not hold for nontrivial scattering amplitudes of the bosonic string. It also does not hold for type-II or heterotic strings.

(b) Holds for any scattering amplitude in the bosonic string. In a modified form that will be explained below it holds for type-II or heterotic strings after odd moduli have been integrated out.

(c) Holds for type-II or heterotic strings, as was shown in Sec. III.K for exponential insertions. In Sec. III.M it was also shown to hold in detail for the one-loop case. We can argue that it holds for all amplitudes. Note that we do not assume here that Ω and $\chi_{\bar z}{}^+$ are complex coordinates for $s\mathcal{M}_h$.

(d) Holds for the partition function of type-II or heterotic strings, as was shown in Sec. VII.F, on the condition that a factor $\mathrm{sdet}(\mathrm{Im}\hat{\Omega})$ be included in the measure. In fact, it follows from (c) as we shall show below.

(e) Holds provided we can argue—as we will indeed—that Ω and $\chi_{\bar z}{}^+$ of (c) are complex coordinates for $s\mathcal{M}_h$. In that case it is equivalent to (c), and valid for all amplitudes in type-II or heterotic strings. Thus it is the properties (c) and (e) which provide the correct framework for superstring perturbation theory.

In the remainder of this section, we shall show that (c) implies (e) and finally see how odd moduli can be integrated out to obtain a result of type (b) for the superstring.

2. Supersymmetric period matrix

Our starting point is an arbitrary scattering amplitude at fixed momenta, encountered already in Sec. III.K, and more explicitly in Eq. (3.323) for type-II and Eq. (3.326) for heterotic strings. We shall mostly be interested in the matter part,

$$\mathcal{A}_m(\Omega,\bar\Omega;\chi_{\bar z}{}^+;\chi_z{}^-;\zeta_i,\bar\zeta_i;p_I^\mu)=\int D(x\psi)\prod_{\mu,I}\delta\left[\oint_{A_I}dz\,\partial_z x^\mu-p_I^\mu\right]V_1(\zeta_1,\bar\zeta_1)\cdots V_n(\zeta_n,\bar\zeta_n)e^{-I_m}\ , \tag{7.111}$$

where the emission vertices $V_1\cdots V_n$ are physical and independent of the ghost fields. For simplicity, we shall consider only the case of even-spin structure, and we shall list the modifications resulting from odd-spin structure at the end.

It is easy to see that $\mathcal{A}_m$, defined above and for all internal momenta, is invariant under local reparametrizations (connected to the identity) and local supersymmetry and has the standard Weyl and U(1) anomalies, which should be thought of as compensated by the ghost fields. $\mathcal{A}_m$ fails to be modular invariant because we picked a canonical homology basis. Thus it may be expected to transform "covariantly" under a modular transformation, provided χ is transformed appropriately. It is also invariant under any large diffeomorphism that preserves the homology basis and hence is invariant under the Torelli group. The most important thing here is that it is reparametrization and supersymmetry invariant.

Now the chiral splitting established in Sec. III.K implies that this is the norm square of a function $\mathcal{C}_v$ dependent only on Ω, $\chi_{\bar z}{}^+$, ζ_i, and p_I^μ:

$$\mathcal{A}_m(\Omega,\bar\Omega;\chi_{\bar z}{}^+,\chi_z{}^-;\zeta_i,\bar\zeta_i;p_I^\mu)=(2\pi)^{10}\delta(k)\,|\,\mathcal{C}_v(\Omega,\chi_{\bar z}{}^+,\zeta_i;p_I^\mu)\,|^2\ . \tag{7.112}$$

Here $\mathcal{C}_v$ inherits[42] the symmetries of $\mathcal{A}_m$. But we know most of the p dependence on $\mathcal{C}_v$: it is a Gaussian in p. A particularly interesting quantity is the variance of the Gaussian:

$$\delta^{\mu\nu}\hat\Omega_{IJ}=\frac{1}{2\pi i}\frac{\partial^2}{\partial p_I^\mu\partial p_J^\nu}\ln\mathcal{C}_v(\Omega,\chi_{\bar z}{}^+,\zeta_i;p_I^\mu)\ . \tag{7.113}$$

Using the functional integral representation of Eq. (3.201), we see that it is independent of the external momenta k, and we get

$$\hat\Omega_{IJ}=\Omega_{IJ}+\frac{1}{2\pi id}\frac{\partial^2}{\partial p_I^\mu\partial p_J^\mu}\ln\int D\psi_+^\mu e^{-I_{\psi_+}+\mathcal{L}'_++2\pi p_I^\mu\sigma_I^\mu} \tag{7.114}$$

with the field σ given as before,

$$\sigma_I^\mu=\frac{1}{4\pi}\int d^2z\,\chi_{\bar z}{}^+(z)\psi_+^\mu(z)\omega_I(z)\ .$$

Since the ψ_+ integral is again Gaussian in p, $\hat\Omega$ is actually independent of p as well and depends only on the supermoduli. The term $\mathcal{L}'_+$ introduces the coupling to the Dirac field of a nonlocal potential (since we have already integrated out the x field), as can be seen from Eq. (3.189) directly. Thus it is appropriate to introduce a full Dirac propagator $\hat S_v(z,w)$ for the combinations of I_{ψ_+} and $\mathcal{L}'_+$:

$$\partial_{\bar z}\hat S_v(z,w)+\frac{1}{8\pi}\chi_{\bar z}{}^+(z)\int d^2z'\chi_{\bar z}{}^+(z')\partial_z\partial_{z'}\ln E(z,z')\times\hat S_v(z',w)=2\pi\delta^2(z,w)\ . \tag{7.115}$$

With the help of this propagator, we have

$$\hat\Omega_{IJ}=\Omega_{IJ}-2\pi i\langle\sigma_I\sigma_J\rangle_{\hat S} \tag{7.116}$$

or

$$\hat\Omega_{IJ}=\Omega_{IJ}-\frac{i}{8\pi}\int d^2z\int d^2w\,\omega_I(z)\chi_{\bar z}{}^+(z)\hat S_v(z,w)\times\chi_{\bar z}{}^+(w)\omega_J(w)\ . \tag{7.117}$$

Reparametrizations, Weyl, and local U(1) invariance of $\hat\Omega$ are manifest, but since $\mathcal{C}_v$ was also supersymmetric, we conclude that $\hat\Omega$ must be supersymmetric.[43]

Thus chiral splitting has led to a supersymmetric extension of the period matrix—exactly the type of thing we were looking for, as will become clear shortly. The imaginary part of this *supersymmetric period matrix* was already encountered in the discussion of the superholomorphic splitting of the superdeterminant on scalar superfields (not surprisingly) in Sec. VII.F for $n=0$,

$$\langle\Phi_I\,|\,\Phi_J\rangle=\mathrm{Im}\hat\Omega_{IJ}\ , \tag{7.118}$$

with $\Phi_{I,J}\in\mathrm{Ker}\mathcal{D}_-^{1/2}$. Recall that, since we are dealing with even-spin structure only, there are precisely h such super Abelian differentials. For odd-spin structure, there is (generically) one more.

In fact, this shows us right away, when we integrate over the internal momenta, that the correct normalization of the matter functional integrals is sdet(Im$\hat\Omega$), as

[42]An arbitrary (real) phase that could in principle come in the definition of $\mathcal{C}_v$ cannot depend on the complex variables Ω, $\chi_{\bar z}{}^+$, and ζ_i, and can thus be neglected—in particular in symmetry considerations. From one spin structure to another it is determined by the action of the modular group.

[43]Since the propagator $\hat S$ has a perturbative series in $\chi_{\bar z}{}^+$ that ends after h terms, Eq. (7.117) is the full answer for $\hat S$ replaced by S when $h=2$. It is straightforward to show directly in that case that $\hat\Omega$ is supersymmetric.

opposed to det(ImΩ), recovering (for the amplitudes with no vertex insertions) that

$$\frac{\text{sdet}\mathcal{D}_+\mathcal{D}_-^{(0)}}{\int d^2\mathbf{z}\,E\;\text{sdet}\,\text{Im}\hat{\Omega}}$$

indeed factors into the absolute value square of a function dependent only on Ω and $\chi_{\bar{z}}{}^+$. This very strongly suggests that Ω and $\chi_{\bar{z}}{}^+$ should be good holomorphic coordinates for supermoduli space. Actually, this may be seen very directly from the fact that $H_-{}^z$ of Eq. (7.83) was a good complex coordinate for supermoduli space and that its expression in Wess-Zumino gauge [Eq. (3.129)] indicates that $e_{\bar{z}}{}^m\delta e_m{}^z$ and $\delta\chi_{\bar{z}}{}^+$ are both good complex coordinates. Hence we have shown that chiral splitting of (c) implies the superholomorphic splitting of (d) for the partition function as well as (e), since now Ω and $\chi_{\bar{z}}{}^+$ are good coordinates for $s\mathcal{M}_h$. As indicated before, this means that (c) and (e) are in fact equivalent.

Actually, the above construction of the supersymmetric period matrix is equivalent to a generalization of the usual construction of the period matrix in terms of line integrals of Abelian differentials. To see this, recall that a holomorphic super Abelian differential $\varpi=\varpi_0+\theta\hat{\omega}+(i/4)\theta\bar{\theta}A\varpi_0$ satisfies the following set of differential equations:

$$D_{\bar{z}}\varpi_0+\tfrac{1}{2}\chi_{\bar{z}}{}^+\hat{\omega}=0\;,$$
$$D_{\bar{z}}\hat{\omega}+\tfrac{1}{2}D_z(\chi_{\bar{z}}{}^+\varpi_0)=0\;. \tag{7.119}$$

The general solution to the second equation is given in terms of h complex integration constants c_I,

$$\hat{\omega}(z)=\sum_I c_I\omega_I(z)-\frac{i}{4\pi}\int d^2w\;\partial_z\partial_w G(z,w)\chi_{\bar{w}}{}^+\varpi_0(w)\;, \tag{7.120}$$

where G is the scalar Green's function. Using the fact that

$$\partial_z\partial_w G(z,w)=-\partial_z\partial_w \ln E(z,w)+\pi\omega_I(z)(\text{Im}\Omega)^{-1}_{IJ}\omega_J(w)\;,$$

we see that the latter contribution can be lumped together with the integration constants c_I. In view of the fact that the prime form is single valued around A cycles, it is then clear that the differentials

$$\hat{\omega}_I(z)=\omega_I(z)-\frac{1}{16\pi^2}\int d^2y\int d^2w\;\partial_z\partial_w \ln E(z,w)\times\chi_{\bar{w}}{}^+S_\nu(w,y)\chi_{\bar{y}}{}^+\hat{\omega}_I(y) \tag{7.121}$$

are canonically normalized around A cycles,

$$\oint_{A_K} d\mathbf{z}\,\varpi_J=\oint_{A_K} dz\,\hat{\omega}_J=\delta_{JK}\;.$$

Here $S_\nu(z,w)$ denotes the meromorphic Dirac propagator [the Szegö kernel for even-spin structure and the propagator (3.204) for odd-spin structure]. The integral of these normalized differentials around B cycles reproduces precisely the supersymmetric period matrix defined above:

$$\hat{\Omega}_{JK}=\oint_{B_K}\varpi_J=\Omega_{JK}-\frac{i}{8\pi}\int d^2w\int d^2y\;\omega_K(w)\chi_{\bar{w}}{}^+S_\nu(w,y)\times\chi_{\bar{y}}{}^+\hat{\omega}_J(y)\;, \tag{7.122}$$

as is easily seen, order by order, in an expansion in powers of χ.

The supersymmetric period matrix in the context of Eq. (7.118) was first encountered in the general formulation of amplitudes in terms of two-dimensional supergeometry in Sec. III.I, and in D'Hoker and Phong (1987a). The construction in terms of line integrals around closed contours of Abelian differentials for even spin structure is due to Bershadsky (1988) and Sonoda (1987d, 1987e). Its supersymmetry was also checked explicitly in Sonoda (1987e). Generalizations to the case of odd-spin structures are given in D'Hoker and Phong (1988a).

3. Splitting of supermoduli space over "moduli"

It remains to work out how the split expressions of (c) and (e) can be reduced to the holomorphic splitting in the sense of (b). In short, we should integrate out the odd moduli.

If $s\mathcal{M}_h$ were a vector bundle above $\mathcal{M}_h$, with the odd moduli as fibers and transition functions that depend only on $\mathcal{M}_h$, then such integration would be straightforward. However, supermoduli space rather emerges as a coset space of two-dimensional supergeometries by reparametrizations, supersymmetry, local U(1), and Weyl transformations. Especially supersymmetry is very tricky, since its action on the two-dimensional metric is not only along the reparametrization and Weyl directions, but also along moduli. Thus changes in $\chi_{\bar{z}}{}^+$ viewed as supersymmetry transformations can be undone only at the expense of a simultaneous motion on moduli space. If one indeed wants to exhibit a projection from any slice of supergeometry for supermoduli space, taken as some specific choice of $e_m{}^a,\chi_{\bar{z}}{}^+$, one has to confront the problem that supersymmetry acts by mixed transformations on both the zweibein and the gravitino field. In practice it has not appeared to be possible in general to disentangle this action and decompose it onto reparametrizations and Weyl transformations without affecting moduli. This observation may lead one to believe that no natural projection of supermoduli onto moduli exists in general.

When this is the case, the formulas for the amplitudes of superstring scattering processes seem ambiguous because, without a preferred projection, the answer for physical amplitudes should be independent of the projection. Actual calculations show that this is not the case: a difference in projection produces a shift in the even coordinates that depends on the odd moduli and, upon integrating out the odd moduli, results in a total derivative term on moduli space.

In fact the emergence of total derivative terms is directly observed when performing a change of slice for the super Beltrami differentials in Eq. (3.335). It was argued by Verlinde and Verlinde (1987b) that a change in $\chi_{\bar z}{}^{+}$ induces a BRST change that may be pulled out of the integral and as usual produces a total derivative on moduli space. Actually, their argument is only local on moduli space, so that a change in $\chi_{\bar z}{}^{+}$ produces a total derivative on moduli within the open patch one is considering, and the question arises how to put such patches together. To be more precise, the super Beltrami differentials are characterized by points z_a, which should move independently of moduli if the proposed formula (3.340) in terms of picture-changing operators is to hold. Thus the issue is whether one can have points moving quasiconformally on moduli space in a global way. Reformulated in terms of the Teichmüller universal curve, it is a question of whether there are any covariantly constant global sections of this fiber bundle. Certainly this bundle is not flat, since we evaluated its nonzero characteristic class c_1. It also has no global sections. It would thus appear that the fermionic string integral is intrinsically ambiguous.

One is faced very much with a problem in Čech cohomology, as it appears perhaps most simply in the problem of the magnetic monopole inside a sphere. One has an object (say the field strength) that is a total derivative (say of the vector potential) in an open patch. However, if one is dealing with an underlying topologically nontrivial manifold, the integral can still be nonzero because one can never cover that manifold with just one patch. Correct expressions must also include the Wu-Yang-type corrections that take the effects of patch changing into account.

Such a treatment was proposed by Verlinde (1987) and independently by the authors, and it leads to a well-defined expression for the full amplitudes, with no further total derivative ambiguities. Thus in general there are additional contributions coming from the boundary terms in a cell decomposition of moduli space, which may be evaluated explicitly. It is tempting to propose that such a treatment could be obtained directly from an argument based on the preservation of worldsheet supersymmetry, but we shall not explore this possibility further here.

To make contact with the discussion given above, we could, for example, consider the chiral amplitude $\mathcal{C}_v$. It depends on Ω and $\chi_{\bar z}{}^{+}$, which were argued to be good complex coordinates for supermoduli space. However, they exhibit the same problem mentioned above: $\chi_{\bar z}{}^{+}$ transforms simply under a local supersymmetry, but Ω_{IJ} also transforms. Thus it seems that we cannot expect to integrate out $\chi_{\bar z}{}^{+}$ and be left with a sensible theory on moduli space in terms of Ω_{IJ}, which is not supersymmetry.

Here, however, we are saved by the existence of $\hat\Omega$, which is supersymmetric. Indeed, it is clear that the amplitude $\mathcal{C}_v$ can be expressed as a function of $\hat\Omega$ and $\chi_{\bar z}{}^{+}$ instead of Ω and $\chi_{\bar z}{}^{+}$:

$$\hat{\mathcal{C}}_v(\hat\Omega,\chi_{\bar z}{}^{+},\zeta_i;p_I^{\mu})=\mathcal{C}_v(\Omega,\chi_{\bar z}{}^{+},\zeta_i;p_I^{\mu}) \ .$$

Since $\hat C_v$ is itself supersymmetric in the sense that polarization tensor and position of vertex operators transform covariantly under supersymmetry, it is clear that we are no longer concerned by the fact that no global slice can be chosen for the super Beltrami differentials.

In fact, for all practical purposes, $(\hat\Omega,\chi)$ admits a natural projection to $\hat\Omega$ trivially defined by omitting χ. Thus, in order to integrate out the odd moduli in a supersymmetric fashion, one should keep $\hat\Omega$ fixed and integrate the remaining independent variable χ. This is not to say that χ suddenly admits global sections above moduli space, but rather that a change of section (a gauge transformation—in this case a supersymmetry) acts in a tensorial fashion, so that, upon transition from one patch to the next, quantities transform in a tensorial way, and no boundary problems occur between patches. In this way we obtain a well-defined measure on "moduli space," viewed as the space of matrices $\hat\Omega$.

4. Modular invariance

We can now argue that the superstring measure in terms of $\hat\Omega$, as prescribed above with the odd moduli integrated out, is modular invariant. It may be convenient to review here the points of the previous discussion that we shall need in our arguments. The first important fact is that

(a) $\hat\Omega$ transforms under modular transformations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

exactly the same way as Ω. This is most easily seen from the description (7.122) of $\hat\Omega$ in terms of line integrals of super Abelian differentials over homology cycles, since a modular transformation is just a change of homology basis. The theta characteristics $[\delta]$ of the $\frac{3}{2}$ differentials χ_a change accordingly,

$$[\delta]\to\begin{bmatrix} D & -C \\ -B & A \end{bmatrix}[\delta]+\frac{1}{2}\begin{bmatrix}\operatorname{diag}CD^t \\ \operatorname{diag}AB^t\end{bmatrix} . \qquad (7.123)$$

The next outcome of our earlier discussions is that

(b) the superstring measure on $\hat\Omega$ resulting from integrating out odd moduli is invariant under small changes of the $2h-2$ $\frac{3}{2}$ differentials χ_a which leave $\hat\Omega$ fixed. In fact the chirally symmetric superdeterminants are regularized in a manifestly super-reparametrization-invariant way, and although chiral splitting of each of them would lead to anomalies, the anomalies cancel in the full gauge-fixed superstring, as we saw in Sec. VII.F. This means that there is no local supersymmetry anomaly, and our assertion follows from the fact that a change in χ_a with $\hat{\Omega}$ fixed is just a supersymmetry transformation. The final point we wish to make is that

(c) modular transformations can be used to pull back or push forward measures on $[\hat{\Omega},\chi]$ without ambiguities. To see why, we note that measures written in terms of $(\hat{\Omega},\chi)$ involve combinations of chiral Dirac determinants, correlation functions of spinors, and correlation functions of scalars. There are no difficulties with spinor correlation functions, but Dirac determinants and chiral scalars in general cannot be defined individually in a modular-invariant way. In our situation, however, we know that modular anomalies will cancel for the combination of Dirac determinants alone, as can be checked from the explicit bosonization formulas (7.61), or by invoking Witten's (1985b) result on global anomalies. As for chiral scalars, the fixed internal momenta splitting prescription applies, which transforms under modular transformations as it should.

We can now see that there is no ambiguity in the superstring measure. More precisely, we can cover the space of $\{\hat{\Omega}\}$ by patches $\{B_\alpha\}$ over each of which a choice of $2h-2$ $\frac{3}{2}$ differentials $\{\chi_{a,\alpha}\}$ is made, and the superstring measure is obtained by expressing χ as $\chi=\sum m_a \chi_{a,\alpha}$ and integrating with respect to $\prod_{a=1}^{2h-2} dm_a$. Over overlaps $B_\alpha \cap B_\beta$ there is no ambiguity in view of observation (b) above. Now let B_α and B_γ be patches for which there exists a modular transformation M sending B_α into MB_α with a nonempty intersection with B_γ. For the superstring measure to be consistent, we need to know that the superstring measure on B_α is pushed to a measure on MB_α that agrees with the measure on B_γ chosen independently at the outset. Under a modular transformation, the measure on B_α is pushed by (c) to a measure of the same functional form, with the only difference that the χ_a from the push forward in general will not agree with the $\chi_{a,\gamma}$ on B_γ. In view of observation (b), this leaves the measure unchanged, and we have shown the absence of modular anomalies.

We can now trace easily the origin of the ambiguities discussed by Verlinde (1987), Verlinde and Verlinde (1987b), Atick, Rabin, and Sen (1988), and Moore and Morozov (1988). These ambiguities seem to be inherent in a choice of slice in which the zweibein $e_m{}^a$ is independent of odd moduli. In this case a change of χ_a keeping $e_m{}^a$ fixed is not a supersymmetry transformation, and the difference in the χ_a results in a total derivative defined only on intersections of small patches on moduli space. The argument we just gave above then fails, since the measure pushed forward on MB_α differs from that on B_γ by a local total derivative. This is why Wu-Yang terms have to be introduced by hand to lead to a well-defined cosmological constant. We also note that, if there existed global sections of the universal Teichmüller curve, so that the $\{\chi_a\}$ could be chosen globally to be invariant under modular transformations, then the above argument would apply trivially. Indeed the measure pushed forward on MB_α would clearly agree with the one on B_γ since they would both come from the same choice of χ_a. However, the Teichmüller curve has no global sections, and Wu-Yang terms will be needed. They are usually difficult to evaluate explicitly.

We observe that the issue of modular invariance, which is a global issue, has been reduced to local considerations by the above arguments. The reason for this is that we already know how to cancel modular anomalies in the chiral Dirac determinants and how to define chiral scalars, using internal loop momenta. The main problem at this point is really the problem of making small changes in the χ_a, which is solved by using the supersymmetric period matrix. In particular, we make no assumption about global choices of χ_a's through the $\hat{\Omega}$ space and just use small covering patches.

As we just noted, slices $[\hat{\Omega},\chi]$ correspond to zweibeins depending usually on odd moduli m_a. This means that the terms $\partial/\partial m_a$ arising from the $\prod dm_a$ integration cannot be dropped. In principle we should expand the contractions in χ, which will stop after $h-1$ terms.

The supersymmetric period matrix $\hat{\Omega}$ will be an element of Siegel space of $h\times h$ symmetric matrices with even Grassmann values. Such a space will have dimensions $\frac{1}{2}h(h+1)$, evidently larger than the dimension $3h-3$ of the space of superperiod matrices. It is obviously an important issue in the present approach to solve the corresponding Schottky problem of characterizing the supersymmetric period matrices arising from super-Riemann surfaces.

5. The cosmological constant to two loops

That these ideas make sense is easily seen by reconsidering some of the calculations performed in the literature. In Morozov and Perelomov (1987) and Atick, Rabin, and Sen (1988) it was argued that, in order to make the string measure well behaved, the insertions of the picture-changing operators in the case of genus 2 should be taken at special points. It is now easy to see why by examining the difference between $\hat{\Omega}$ and Ω in Eq. (7.117). When $h=2$, we can replace $\hat{S}_\nu$ by S_ν, which is the Szegö kernel for genus 2. Furthermore, $\chi_{\bar{z}}{}^+$ is given by

$$\chi_{\bar{z}}{}^+(z)=\alpha_1\delta(z-z_1)+\alpha_2\delta(z-z_2)\ , \qquad (7.124)$$

where z_1 and z_2 are two arbitrary points and α_1 and α_2 are the two odd moduli. Substitution into Eq. (7.117) shows that only the term in $\alpha_1\alpha_2$ survives, so that the Szegö kernel is evaluated between z_1 and z_2,

$$\hat{\Omega}_{IJ}=\Omega_{IJ}-\frac{i}{4\pi}\alpha_1\alpha_2\omega_I(z_1)S_\nu(z_1,z_2)\omega_J(z_2)\ . \qquad (7.125)$$

An explicit formula for the Szegö kernel in terms of branch points was given by Fay (1977):

$$S_\nu(x,y)=\frac{1}{2}\left[\left[\frac{\psi(x)}{\psi(y)}\right]^{1/4}+\left[\frac{\psi(y)}{\psi(x)}\right]^{1/4}\right]\times\frac{1}{z(x)-z(y)}\left[\frac{\partial z(x)}{\partial x}\frac{\partial z(y)}{\partial y}\right]^{1/2} \quad (7.126)$$

with

$$\psi(x)=\frac{\prod_{a_i\in A}[z(x)-a_i]}{\prod_{a_i\in B}[z(x)-a_i]} \quad (7.127)$$

in the notation of Sec. IV.B, so that a_i, $i=1,\ldots,6$ are the branch points and $A\cup B$ is the partition of the branch points into two groups of 3 corresponding to the spin structure ν. This formula shows that the divisor in y of $S_\nu(a_1,y)$ is

$$\sum_A a_i - 2a_1$$

and similarly if $a_1\in B$, so that one is left with $\hat{\Omega}=\Omega$ when z_1 and z_2 are two branch points within either A or B.

Thus it is to be expected that the correct superstring measure can be written in terms of the usual measure in Ω when the support of χ is located at the branch points. This also includes the calculation of the cosmological constant to two-loop order by Moore and Morozov (1987), also performed with the insertions at the branch points.

We have, however, the general supersymmetry covariant formula available, and hence we can insert the picture-changing operators anywhere. When they are not inserted at the branch points, Ω will not equal $\hat{\Omega}$, and the difference is an odd-moduli-dependent shift, which according to general integration formulas for Grassmann variables will produce a total derivative term on moduli space. The difference here, however, is that this contribution was defined tensorially throughout, so that only a term coming from the boundary of moduli space and not from the boundary of individual cells is obtained. We shall report elsewhere on more explicit verifications of these ideas.

The importance of a modular-invariant choice of insertions z_a is stressed in Verlinde and Verlinde (1987b). Ambiguities caused by total derivatives on local patches of moduli space were investigated by Verlinde (1987), Atick, Rabin, and Sen (1988), and Moore and Morozov (1988). Appropriate corrections to the cosmological constant dictated by Čech cohomology considerations (as in Wu-Yang terms for a particle in a gauge field; see Alvarez, 1985) were introduced by Verlinde (1987) and in unpublished work of D'Hoker and Phong (1987). A different approach assuming the existence of global sections of the universal Teichmüller curve is to be found in Atick, Moore, and Sen (1988a, 1988b) where a discussion of advantages and disadvantages of various choices of slices in the literature is also given. The possibility that additional contributions from the boundary of moduli space may be required in the covariant or light-cone gauge superstring in view of supersymmetry has been suggested by Green and Seiberg (1987) and Greensite and Klinkhamer (1987). Detailed analyses of contributions from the boundary of moduli space are presented by Atick and Sen (1987a, 1987b), who show that two-loop string-theoretic calculations of Fayet-Iliopoulos D terms agree with the effective field considerations of Dine, Ichinose, and Seiberg (1987), and Dine, Seiberg, and Witten (1988); Atick, Moore, and Sen (1988b) also address ambiguities of n-point functions.

VIII. VERTEX OPERATORS FOR ON-SHELL PHYSICAL PARTICLES

One of the remarkable features of string theories is that correlation functions of certain local operators—the vertex operators—on the worldsheet give scattering amplitudes of physical particles in space-time. The spectrum of the space-time theory as well as its gauge invariances are thus dictated by the structure of those vertex operators. The general rules for the construction of vertex operators have been partially known from the days of dual models. A key requirement is that they have conformal dimension 1 for open strings and (1,1) for closed strings. In this section we shall present the complete rules for vertex operators in the functional formulation for the closed bosonic, type-II, and heterotic string theories. Essentially, bosonic vertex operators must be consistent with all the symmetries of the corresponding worldsheet theory, after inclusion of all possible anomalies. The condition of conformal dimension (1,1) just guarantees the integrability of a vertex operator on the worldsheet. Of particular importance is the Weyl anomaly. It is in fact the anomalous dimension of vertex operators that is responsible for the appearance of massless spin-2 particles in the string spectrum. Vertex operators also give a simple explanation of gauge invariances in space-time as modifications by total derivatives on the worldsheet, since these should not change the scattering amplitudes. We shall discuss in some detail the example of the gauge symmetry of the graviton and antisymmetric tensor field.

Vertex operators for emission of fermions are more complicated. Some of the difficulties can already be gathered from the fact that we must manufacture space-time spinors when the fundamental fields on the worldsheet are space-time vectors. Moreover, insertion of a fermion emission vertex operator should change the spin structure on the worldsheet. In Sec. VIII.E we shall present the fermion vertex construction due to Friedan, Shenker, and Martinec (1985; Friedan, Martinec, and Shenker, 1986) and Knizhnik (1985) based on bosonization and coupling to the ghosts. The space-time supersymmetry charge is then easily obtained from the fermion vertex operator, and some basic consequences of supersymmetry will be discussed.

A. Covariance properties of vertex operators

Vertex operators for on-shell physical states of given momentum k must obey the following covariance properties.

(i) Space-time translation invariance requires that all x^μ dependence occur through a factor of $\exp(ik\cdot x)$. The remaining factors depend only on the derivatives of x^μ.

(ii) Space-time Lorentz invariance requires that space-time indices $(\mu,\nu,\ldots)$ on all fields be contracted with a polarization tensor $\varepsilon_{\mu\nu}(k)$ which transforms under a real representation of the little group of k_μ.

(iii) Worldsheet reparametrization invariance is ensured when Einstein indices are contracted with the zweibein to yield U(1) indices. A factor $\sqrt{g}=\det e_m{}^a$ is required for the volume element.

(iv) Worldsheet local U(1) invariance requires that derivatives be covariant, all U(1) indices properly contracted, and all U(1) anomalies canceled.

(v) Weyl invariance requires that the vertex be invariant under Weyl rescalings after inclusion of all anomalies.

Fermionic strings require, in addition to the above, the following.

(vi) Local worldsheet supersymmetry. Vertices must be invariant under arbitrary reparametrizations of superspace ($N=1$ for the type-II superstring, $N=\frac{1}{2}$ for heterotic strings). With superfields, all super Einstein indices must be contracted with the superzweibein, only local U(1) covariant derivatives should be used, and a factor $E=\operatorname{sdet}E_M{}^A$ should be included instead of $\sqrt{g}$.

(vii) Super Weyl invariance must be preserved after inclusion of anomalies.

Now requirements (i)–(iii) and (vi) are easily enforced by use of U(1) covariant (super)derivatives, while (vii) will follow from (v) and (vi). Further, there will be no U(1) anomaly if Weyl anomalies cancel separately for left- and right-movers. Thus (super)Weyl invariance is the key property that distinguishes physical states from ghost states.

B. The bosonic string and space-time gauge invariance

1. The bosonic string vertex operators

The general vertex operator consistent with the requirements of Sec. VIII.A is given by

$$V(\varepsilon,k)=\int_M d^3\xi\sqrt{g}\,U(\varepsilon,Dx,R)e^{ik\cdot x} . \tag{8.1}$$

Here U is a polynomial scalar expression in the U(1) covariant derivatives of x^μ and the two-dimensional curvature R.[44] Using the Heisenberg equations of motion for the x^μ field ($D_zD_{\bar z}x^\mu=0$) under the time-ordering symbol, we see that vertex operators involving $D_zD_{\bar z}x^\mu$ must be omitted, and on a given x^μ only D_z or $D_{\bar z}$ derivatives are applied. We turn then to the Weyl transformation laws of U(1) covariant derivatives. If $e_m{}^a$ is a zweibein, the connection and curvature are

$$\omega_m=-e_m{}^c\varepsilon^{ab}e_a{}^n e_b{}^p\partial_n e_p{}^c,\quad R=\varepsilon^{mn}\partial_m\omega_n .$$

The covariant derivatives on tensors of U(1) weight n are given by

$$D_z^n=e_z{}^m\nabla_m^n,\quad D_{\bar z}^n=e_{\bar z}{}^m\nabla_m^n,\quad [D_z,D_{\bar z}]^n=nR . \tag{8.2}$$

Under Weyl transformations ($e_m{}^a=e^{\sigma}\hat e_m{}^a$) we have

$$\begin{aligned} D_z^n&=e^{(n-1)\sigma}\hat D{}_z^n e^{-n\sigma} ,\\ D_{\bar z}^n&=e^{-(n+1)\sigma}\hat D{}_{\bar z}^n e^{n\sigma} ,\\ R&=e^{-2\sigma}(\hat R-2\hat D_z\hat D_{\bar z}\sigma) . \end{aligned} \tag{8.3}$$

U(1) invariance of $U(\varepsilon,Dx,R)$ implies that the total number of derivatives—independently of how they are distributed—must satisfy

$$\#D_z=\#D_{\bar z} . \tag{8.4}$$

On the other hand, the possible sources of Weyl anomalies are the following.

(a) Contractions within $\exp(ik\cdot x)$. Under constant Weyl rescalings we have

$$e^{ik\cdot x}\mapsto e^{-\sigma k\cdot k}e^{ik\cdot x} , \tag{8.5}$$

so that in view of the Weyl scalings of the derivative factors we find

$$m^2=-k\cdot k=2(N-1),\quad N=0,1,\ldots . \tag{8.6}$$

Thus, at the lowest mass level, $N=0$, we have a tachyon whose presence has manifested itself in the asymptotic behavior of the string partition function (cf. Secs. II.H, V.F, and VII.A). At the next mass level, $N=1$, we have massless particles. Under the Lorentz group they decompose into the graviton, the dilaton, and the antisymmetric tensor field. It is a remarkable property of string theory that the graviton must invariably be present. (At least in the critical dimension.)

(b) Contractions of D_z derivatives with each other or with $\exp(ik\cdot x)$. The first type produces an anomaly proportional to $\eta_{\mu\nu}$, while the second produces an anomaly proportional to k_μ. Such anomalies disappear when the polarization tensor is made to satisfy

$$k^\mu\varepsilon_{\mu\nu}\ldots=k^\nu\varepsilon_{\mu\nu}\ldots=0 . \tag{8.7}$$

These polarization tensors do not make up the complete list, however, since cancellation of anomalies could occur by combining different terms. The same considerations apply to contractions between $D_{\bar z}$ derivatives.

(c) Contractions between D_z and $D_{\bar z}$ derivatives. These also lead to anomalous terms. However, these mixed

[44]To obtain similarity with the case of fermionic strings, where derivatives are taken U(1) covariant, we have adopted this same strategy for the bosonic case. The translation to the covariant derivatives ∇ introduced previously is straightforward.

contractions always require curvature counterterms, and conversely, since curvature terms by themselves are not Weyl invariant, they can only compensate for mixed derivatives in view of their tensor structure. If we introduce "normal ordering" conventions so that no mixed contractions are to be performed, no curvature terms are required and none will ever appear. We shall throughout assume that such normal ordering has been performed.

Thus the complete classification of vertex operators is contingent upon the evaluation of all contractions of derivatives of x^μ at coincident points. Since the action is quadratic we need only consider bilinear composites in x^μ. The use of the Heisenberg equations of motion under the time-ordered product and the commutation relations of (8.2) allow us to restrict ourselves to the case of no mixed derivatives in z and $\bar{z}$ on a given x^μ. Thus the only contractions of interest are

$$\langle D_z{}^m x(z) D_z{}^n x(z)\rangle \ . \tag{8.8}$$

We shall regularize the ultraviolet behavior of composites by the heat kernel and a finite-time cutoff, which is reparametrization invariant. The results agree with those obtained from reparametrization-invariant (but not translation-invariant) Pauli-Villars regulators to the order we have checked. Dimensional regularization would yield different expressions, which we believe are inconsistent with reparametrization invariance; moreover, this method is well known to have problems with infrared behavior. With any of these methods, Leibnitz's rule is satisfied:

$$\partial\langle \partial^p x(z)\partial^q x(z)\rangle = \langle \partial^{p+1}x(z)\partial^q x(z)\rangle + \langle \partial^p x(z)\partial^{q+1}x(z)\rangle \ . \tag{8.9}$$

Here $\partial=\partial/\partial z$ with z a local conformal coordinate. Thus, in view of Leibnitz's rule, we need only compute the contractions $\langle x(z)\partial^m x(z)\rangle$.

In Appendix A we calculate the general expression for such contractions, and we find

$$\langle x(z)\partial^m x(z)\rangle = \sum_{p=1}^{m} \sum_{m_i=1,2,\ldots}^{m_1+\cdots+m_p=m} A_{m_1,\ldots,m_p} e^{2m\sigma}(\partial^{m_1}e^{-2\sigma})\cdots(\partial^{m_p}e^{-2\sigma}) \ , \tag{8.10}$$

where the A's are finite (rational) coefficients given by

$$A_{m_1,\ldots,m_p} = \frac{1}{m}\int_0^1 dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{p-1}}dt_p \frac{\partial^p}{\partial\tau_1\cdots\tau_p}[\tau_0^{m_1}(\tau_0+\tau_1)^{m_2}\cdots(\tau_0+\cdots+\tau_{p-1})^{m_p}(\tau_0+\cdots+\tau_p)^{-m}] \ . \tag{8.11}$$

The rule here is to differentiate with respect to the τ's first and then to set $\tau_0=1-t_1$, $\tau_1=t_1-t_2,\ldots,\tau_{p-1}=t_{p-1}-t_p$, and $\tau_p=t_p$. By performing only the last two differentiations with respect to τ and only the last integration, we may easily obtain a recursion formula ($m=m_1+\cdots+m_p$):

$$\begin{aligned} A_{m_1m_2\cdots m_p} = &- A_{m_1\cdots m_{p-2}(m_{p-1}+m_p)} \\ &+\frac{m+1}{m_p+1}A_{m_1\cdots m_{p-2}(m_{p-1}+m_p+1)} \\ &-\frac{m-m_p}{m_p+1}A_{m_1\cdots m_{p-2}m_{p-1}} \ . \end{aligned}$$

Special cases are

$$A_m = -\frac{1}{m+1} \ ,$$

$$A_{m_1m_2} = \frac{1}{m_1+m_2+1} - \frac{m_1+m_2+1}{m_2+1}\frac{1}{m_1+m_2+2} + \frac{m_1}{m_2+1}\frac{1}{m_1+1} \ .$$

Some low-order terms are easily obtained. In function of covariant derivatives this leads to the following formulas for contractions:

$$\begin{aligned} &\langle xx\rangle = -\frac{2}{\varepsilon}+2\sigma, \quad \langle D_z xx\rangle = D_z\sigma \ , \\ &\langle D_z{}^2 xx\rangle = \tfrac{1}{3}[2D_z{}^2\sigma-(D_z\sigma)^2] \ , \\ &\langle D_z{}^3 xx\rangle = \tfrac{1}{2}(D_z{}^3\sigma-2D_z{}^2\sigma D_z\sigma) \ . \end{aligned} \tag{8.12}$$

Thus contractions of covariant derivatives are polynomials in covariant derivatives of the conformal factor.

Finally, we provide a general argument for equivalence between vertices in the Polyakov formalism and vertices in the operator langauge. Polyakov vertices must be Weyl and reparametrization invariant after anomalies have been taken into account. For the operator vertices, one requires instead that they have conformal weight 1, so that physical states are annihilated by the Virasoro generators L_n for $n\geq 1$. When we put the Polyakov vertex in the conformal gauge, the worldsheet metric is Euclidean and the residual invariance is just the Virasoro invariance. Conversely, every vertex of conformal weight 1 can be lifted to a Weyl-invariant vertex by adding appropriate contractions. This can also be checked explicitly by inspection of the commutators of vertices with the components of the stress tensor.

2. Space-time gauge invariance and examples

The vertex operators constructed by the above methods may not produce truly distinct particle states, since they may differ by a total derivative on the worldsheet providing zero upon integration. In space-time this would correspond to a gauge transformation. As an example consider the $m^2=0$ vertex

$$V_0 = \int d^2\xi\sqrt{g}\ \varepsilon_{\mu\bar{\mu}}D_z x^\mu D_{\bar{z}} x^{\bar{\mu}} e^{ik\cdot x} \ . \tag{8.13}$$

We may add to V_0 expressions of the following form, which reduce to zero through integrations by parts and the (Heisenberg) equations of motion:

$$\varepsilon_{\mu\bar{\mu}} \mapsto \varepsilon_{\mu\bar{\mu}} + \theta^1_\mu k_{\bar{\mu}} + \theta^2_{\bar{\mu}} k_\mu + \theta^3 k_\mu k_{\bar{\mu}} \ . \tag{8.14}$$

Clearly, in terms of ε, this precisely corresponds to the gauge transformations associated with the graviton and the antisymmetric tensor field. For massless particles, Eq. (8.14) generates all gauge transformations. To obtain gauge transformations for higher-mass particles it suffices to write down all worldsheet derivatives with the appropriate U(1) structure. A somewhat more complicated example may be given for mass level $m^2=2$, where

$$\begin{aligned} V_1 = \int d^2\xi \sqrt{g}\, (&\varepsilon_{\mu\nu,\bar{\mu}\bar{\nu}} D_z x^\mu D_z x^\nu D_{\bar{z}} x^{\bar{\mu}} D_{\bar{z}} x^{\bar{\nu}} \\ &+ \varepsilon_{\mu\nu,\bar{\mu}} D_z x^\mu D_z x^\nu D_{\bar{z}}{}^2 x^{\bar{\mu}} \\ &+ \varepsilon_{\mu,\bar{\mu}\bar{\nu}} D_z{}^2 x^\mu D_{\bar{z}} x^{\bar{\mu}} D_{\bar{z}} x^{\bar{\nu}} \\ &+ \varepsilon_{\mu,\bar{\mu}} D_z{}^2 x^\mu D_{\bar{z}}{}^2 x^{\bar{\mu}}) e^{ik\cdot x} \ . \end{aligned} \tag{8.15}$$

The conditions for Weyl invariance are easily obtained with the rules for contractions given above (recall that our vertex was implicitly normal ordered with respect to mixed contractions, so that no curvature terms occur),

$$\begin{aligned} (k^2\eta^{\mu\nu} - 6k^\mu k^\nu)\varepsilon_{\mu\nu,\bar{\mu}\bar{\nu}} &= ik^2 k^\mu \varepsilon_{\mu,\bar{\mu}\bar{\nu}} \ , \\ (k^2\eta^{\mu\nu} - 6k^\mu k^\nu)\varepsilon_{\mu\nu,\bar{\mu}} &= ik^2 k^\mu \varepsilon_{\mu,\bar{\mu}} \ , \end{aligned} \tag{8.16}$$

and the analogous conditions with $\mu \to \bar{\mu}$, etc. These constraints are trivially satisfied when the ε's are transverse and traceless, but there are more solutions.

In particular, there is a gauge invariance obtained by adding total derivatives of the type

$$\begin{aligned} 0 = \int d^2\xi \sqrt{g}\, [&\theta^1_{\mu,\bar{\mu}\bar{\nu}} D_z (D_z x^\mu D_{\bar{z}} x^{\bar{\mu}} D_{\bar{z}} x^{\bar{\nu}} e^{ik\cdot x}) \\ &+ \theta^2_{\mu\nu,\bar{\mu}} D_{\bar{z}} (D_z x^\mu D_z x^\nu D_{\bar{z}} x^{\bar{\nu}} e^{ik\cdot x})] \ , \end{aligned}$$

inducing the gauge transformations

$$\begin{aligned} \varepsilon_{\mu\nu,\bar{\mu}\bar{\nu}} &\to \varepsilon_{\mu\nu,\bar{\mu}\bar{\nu}} + \frac{i}{2}(k_\mu \theta^1_{\nu,\bar{\mu}\bar{\nu}} + k_\nu \theta^1_{\mu,\bar{\mu}\bar{\nu}} + k_{\bar{\mu}} \theta^2_{\mu\nu,\bar{\nu}} \\ &\qquad + k_{\bar{\nu}} \theta^2_{\mu\nu,\bar{\mu}}) \ , \\ \varepsilon_{\mu\nu,\bar{\mu}} &\to \varepsilon_{\mu\nu,\bar{\mu}} + \theta^2_{\mu\nu,\bar{\mu}}, \quad \varepsilon_{\mu,\bar{\mu}\bar{\nu}} \to \varepsilon_{\mu,\bar{\mu}\bar{\nu}} + \theta^1_{\mu,\bar{\mu}\bar{\nu}} \ . \end{aligned} \tag{8.17}$$

More details about these examples, as well as the structure of the cancellation of the curvature terms, may be found in D'Hoker and Phong (1987b).

The general rules for vertex operators above were formulated by Weinberg (1985). The contraction $\langle D_x x D_{\bar{z}} x \rangle$ was shown by de Alwis (1986) to lead to the correct dilaton vertex of Fradkin and Tseytlin (1985a, 1985b). Anomalous Weyl scalings of general composites were calculated in D'Hoker and Phong (1987b). Weyl rescalings based on dimensional regularization are to be found in Tanii and Watabiki (1986). Weyl anomalies when several emission points get close to one another are evaluated in Seiberg (1987), Sen (1987), and Watabiki (1987). Vertices from operator considerations are derived in Sasaki and Yamanaka (1985) and Ichinose and Sakita (1986). Indirect methods for obtaining vertex operators from tachyon amplitudes are discussed in Aldazabal *et al.* (1987). Interpretations of gauge invariances as total derivatives are given in Callan and Gan (1986), Cohen *et al.* (1987) and D'Hoker and Phong (1987b).

C. The type-II superstring

Vertex operators for the type-II superstring must be of the form

$$V(\varepsilon,k) = \int d^2\mathbf{z}\, E U(\varepsilon, E_M{}^A, \mathcal{D}X) e^{ik\cdot X} \ , \tag{8.18}$$

where U is a polynomial scalar expression in the U(1) covariant superderivatives of X^μ and of the supercurvature R_{+-}. Super-reparametrization and local U(1) invariance require U to be a scalar under these transformations. Here we use again the Heisenberg equations of motion $\mathcal{D}_+\mathcal{D}_- X^\mu = 0$ under the time-ordering symbol to eliminate mixed derivatives on a single X^μ. Thus the building blocks of U are

(i) $\mathcal{D}_+{}^p X^\mu, \mathcal{D}_-{}^q X^\nu, \quad p \geq 1, \ q \geq 1$,

(ii) R_{+-} and covariant derivatives thereof .

Local U(1) invariance requires the total[45] number of $\mathcal{D}_+$ and the total number of $\mathcal{D}_-$ derivatives to be equal:

$$\#\mathcal{D}_+ = \#\mathcal{D}_- \ . \tag{8.19}$$

Constant super Weyl transformations, including the anomaly of the exponential, require

$$\tfrac{1}{2}\#\mathcal{D}_+ + \tfrac{1}{2}\#\mathcal{D}_- + \#R_{+-} = 1 - k_\mu k^\mu \ . \tag{8.20}$$

Finally, to perform the Gliozzi-Scherk-Olive projection in order to obtain supersymmetry, one must also require that left and right fermion numbers be separately conserved. Including the factors of $d\theta\, d\bar{\theta}$, one gets

$$\#\mathcal{D}_+ + \#R_{+-} = \#\mathcal{D}_- + \#R_{+-} = \text{odd} \ . \tag{8.21}$$

Combining Eqs. (8.19)–(8.21) gives the mass spectrum

$$m^2 = -k_\mu k^\mu = 2N, \quad N = 0,1,2,\ldots \ , \tag{8.22}$$

which is quite familiar from the operator formulation.

Finally, we must insist on local super Weyl invariance, independently for left- and right-movers to ensure local U(1) invariance as well. Contractions of $\mathcal{D}_+{}^n X^\mu$ with $\exp(ik\cdot X)$ and with $\mathcal{D}_+{}^m X^\nu$ will produce factors of covariant derivatives of Σ. Contractions of $\mathcal{D}_+{}^m X^\mu$ with $\mathcal{D}_-{}^n X^\nu$ will always produce curvature terms and covariant derivatives thereof. Assuming that a normal ordering convention has been adopted, so that no mixed derivatives are contracted, there will be no curvature

[45] This total number counts those $\mathcal{D}$'s applied to X^μ or R_{+-} equally.

terms either.

We can now outline the procedure for finding all vertex operators at a given mass level N.

Determine all contractions $\langle \mathcal{D}_+{}^n X \mathcal{D}_+{}^m X \rangle$ with the total number of $\mathcal{D}_+$'s less than or equal to $2N+1$. Consider all expressions of the form (8.18) for the vertex without any curvature terms. Take the polarization tensor to be (anti)symmetric when two powers, say m and n, are equal and (odd) even. Group the terms in Eq. (8.18) into those having an even and those having an odd number of space-time indices. These groups do not mix and can be treated separately. In, say, the even terms, take the tensor of highest weight and separate its traces. Using the results for the contractions between $\mathcal{D}_+$ derivatives, determine successively conditions on the lower weight tensors, so that they combine with the anomalies of the trace of the higher tensors to produce super-Weyl-invariant expressions.

Clearly, the most difficult step in the above procedure is the calculation of the anomalous contractions. As a regulator we again use heat-kernel, short-time cutoff methods, which are guaranteed to be super-reparametrization invariant. The calculation of the contractions can be performed with the help of the super heat kernel constructed in Appendix B (restricted to the very simple case of $n=0$), and we shall just quote the results here. Furthermore, some algebraic relations exist among the different p,q exponents. This comes about because the anomalous contractions satisfy the "derivative property"

$$\mathcal{D}_+\langle \mathcal{D}_+{}^p X \mathcal{D}_+{}^q X\rangle = \langle \mathcal{D}_+{}^{p+1} X \mathcal{D}_+{}^q X\rangle + (-1)^p \langle \mathcal{D}_+{}^p X \mathcal{D}_+{}^{q+1} X\rangle \ , \qquad (8.23)$$

so that it suffices to compute $\langle \mathcal{D}_+{}^p XX\rangle$, the other cases being deduced from it using Eq. (8.23).

Though an explicit formula with known coefficients is not available for the type-II superstring, in contrast to the bosonic string, the calculations are sufficiently tractable to low order, and we get

$$\begin{aligned}
&\langle \mathcal{D}_+ XX\rangle = \mathcal{D}_+\Sigma \ ,\\
&\langle \mathcal{D}_+{}^2 XX\rangle = \mathcal{D}_+{}^2\Sigma \ ,\\
&\langle \mathcal{D}_+{}^3 XX\rangle = \tfrac{1}{2}(\mathcal{D}_+{}^3\Sigma - \mathcal{D}_+\Sigma\mathcal{D}_+{}^2\Sigma) \ ,\\
&\langle \mathcal{D}_+{}^4 XX\rangle = \tfrac{1}{3}(\mathcal{D}_+{}^4\Sigma + \mathcal{D}_+{}^2\Sigma\mathcal{D}_+{}^2\Sigma + \mathcal{D}_+\Sigma\mathcal{D}_+{}^3\Sigma) \ ,\\
&\langle \mathcal{D}_+{}^5 XX\rangle = \tfrac{1}{3}(\mathcal{D}_+{}^5\Sigma - 2\mathcal{D}_+{}^4\Sigma\mathcal{D}_+\Sigma) \ .
\end{aligned} \qquad (8.24)$$

Applying the above rules, we can easily derive the U functions for the lowest mass levels. Certain symmetrization properties that automatically arise in this construction can be usefully represented in terms of Young tableaux of the representations of the target space-time Lorentz group corresponding to the particles of the vertex operator. Thus we have the following.

$m^2=0$:

$$U = \varepsilon_{\mu;\nu}\mathcal{D}_+X^\mu D_- X^\nu \ ,$$

$$k^\mu \varepsilon_{\mu;\nu} = k^\nu \varepsilon_{\mu;\nu} = 0$$

($\varepsilon_{\mu;\nu}$ is *not* traceless, however). Symmetric traceless $\varepsilon_{\mu;\nu}$ corresponds to the graviton; antisymmetric $\varepsilon_{\mu;\nu}$ corresponds to the antisymmetric tensor, and the trace part of $\varepsilon_{\mu;\nu}$ is the dilation (it would require an R_{+-} term if contractions had not already been performed).

$m^2=2$:

$$\begin{aligned}
U = {}&\varepsilon_{\mu\nu\kappa;\lambda\rho\sigma}\mathcal{D}_+X^\mu\mathcal{D}_+X^\nu\mathcal{D}_+X^\kappa\mathcal{D}_-X^\lambda\mathcal{D}_-X^\rho\mathcal{D}_-X^\sigma\\
&+\varepsilon_{\mu\nu;\kappa\lambda}\mathcal{D}_+X^{\{\mu}\mathcal{D}_+{}^2X^{\nu\}}\mathcal{D}_-X^{\{\kappa}\mathcal{D}_-{}^2X^{\lambda\}}\\
&+\varepsilon_{\mu\nu\kappa;\lambda\sigma}\mathcal{D}_+X^\mu\mathcal{D}_+X^\nu\mathcal{D}_+X^\kappa\mathcal{D}_-X^{\{\lambda}\mathcal{D}_-{}^2X^{\sigma\}} \ .
\end{aligned}$$

It is not hard to extend this list to higher-mass levels.

D. The heterotic string

The general vertex operator for the heterotic string is of the form

$$V = \int d^2\xi\, d\theta(\mathrm{sdet}E_M{}^A)U^+e^{ik\cdot x} \ . \qquad (8.25)$$

Since $d\theta(\mathrm{sdet}E_M{}^A)$ is a spinor superfield of weight $\frac{1}{2}$, super-reparametrization and U(1) invariance require that U^+ be a spinor superfield of weight $\frac{1}{2}$, built out of $\mathcal{D}_+{}^pX^\mu$, $\mathcal{D}_{\bar z}{}^qX^\mu$, and $\Psi^I\mathcal{D}_{\bar z}{}^r\Psi^I$, as well as factors of $R_{+\bar z}$ and its covariant derivatives (mixed derivatives on a single factor again have been eliminated though equations of motion). Simple and important examples are the Yang-Mills vertex,

$$V = \varepsilon_\mu^{IJ}\int d^2\xi\, d\theta\, E\mathcal{D}_+X^\mu\Psi^I\Psi^J e^{ik\cdot x} \ , \qquad (8.26)$$

and the gravity multiplet vertex,

$$V = \varepsilon_{\mu\nu}\int d^2\xi\, d\theta\, E\mathcal{D}_+X^\mu\mathcal{D}_{\bar z}X^\nu e^{ik\cdot x} \ . \qquad (8.27)$$

Turning now to the general U^+ we observe that invariance under U(1) and constant Weyl transformations imply, respectively, that

$$\begin{aligned}
&-\tfrac{1}{2}\#\mathcal{D}_+ + \#\mathcal{D}_{\bar z} + \tfrac{1}{2}\#R_{+\bar z} + \tfrac{1}{2}\#\Psi = \tfrac{1}{2} \ ,\\
&\tfrac{1}{2}\#\mathcal{D}_+ + \#\mathcal{D}_{\bar z} + \tfrac{3}{2}\#R_{+\bar z} + \tfrac{1}{2}\#\Psi = \tfrac{3}{2} - k\cdot k \ .
\end{aligned} \qquad (8.28)$$

On the other hand, the vertex must have even worldsheet fermion number, that is, $\#\Psi$ must be even. It follows that

$$m^2 = -k\cdot k = 2N, \ N=0,1,2,\ldots \ . \qquad (8.29)$$

The rules of construction from the principle of super Weyl invariance are now completely analogous to those stated for the type-II superstring once we have identified the potential anomalous contractions. There are new anomalies coming from the contractions $\langle(\mathcal{D}_{\bar z})^p\Psi\Psi\rangle$, while the anomalous contractions for X^μ are of the form $\langle(\mathcal{D}_{\bar z})^pXX\rangle$, $\langle(\mathcal{D}_+)^qXX\rangle$, and $\langle(\mathcal{D}_{\bar z})^pX(\mathcal{D}_+)^qX\rangle$. The first two types of terms involving X^μ, however, are essentially given by those of the bosonic string involving $\mathcal{D}_{\bar z}$ derivatives and those of the type-II superstring involving

$\mathcal{D}_+$ derivatives. To see this, we regulate the theory by the heat-kernel, short-time cutoff method. The natural operator for X^μ in the heterotic string is $\mathcal{D}_+\mathcal{D}_{\bar z}$, which is not a U(1) scalar, so we use instead $-\mathcal{D}_+{}^2\mathcal{D}_{\bar z}$, which is a positive operator transforming as

$$\mathcal{D}_+{}^2\mathcal{D}_{\bar z}=e^{-2\Sigma}(\hat{\mathcal{D}}_+{}^2\hat{\mathcal{D}}_{\bar z}-\hat{\mathcal{D}}_+\Sigma\hat{\mathcal{D}}_+\hat{\mathcal{D}}_{\bar z}) \tag{8.30}$$

under super Weyl scalings. When computing $\langle(\mathcal{D}_+)^p XX\rangle$ we can infer from dimensional analysis and U(1) covariance that p $\mathcal{D}_+$ derivatives acting on Σ will appear in the answer. Omitting then any reference to $\mathcal{D}_-$ in the type-II superstring calculation reduces the operators in the heat-kernel regularization. On the other hand, in an anomaly computation involving only $\mathcal{D}_{\bar z}$ derivatives on Σ, the $\mathcal{D}_+$ derivatives may be effectively omitted, and we recover the result from the bosonic string. Finally, the anomalies in mixed derivatives are polynomials in $R_{+\bar z}$ and its superderivatives. However, as in the bosonic or type-II superstring, such contractions are precisely compensated by the curvature terms, and every curvature term is present only to compensate for the anomalous contractions. Consequently, the normal ordering convention will be adopted in which no mixed contractions are allowed, and thus no curvature terms should appear.

We now discuss anomalies of the spinor superfields Ψ in order to complete our analysis. The basic object is the propagator

$$(\mathcal{D}_+)^{-1}(\mathbf{z},\mathbf{z}')=\langle\Psi(\mathbf{z})\Psi(\mathbf{z}')\rangle\ , \tag{8.31}$$

which in superconformal gauge is related to the flat superspace propagator by

$$(\mathcal{D}_+)^{-1}(\mathbf{z},\mathbf{z}')=e^{-\Sigma(\mathbf{z})/2}(\hat{\mathcal{D}}_+)^{-1}(\mathbf{z},\mathbf{z}')e^{-\Sigma(\mathbf{z}')/2}\ , \tag{8.32}$$

with

$$\hat{\mathcal{D}}_+^{-1}(\mathbf{z},\mathbf{z}')=\frac{z-z'}{|z-z'|^2+\varepsilon^2}+\theta\theta'\delta^2(z-z')\ . \tag{8.33}$$

The natural heat kernel is

$$\mathcal{K}^t_\Sigma(\mathbf{z},\mathbf{z}')=\langle\mathbf{z}\,|\,e^{t2\mathcal{D}_+{}^2\mathcal{D}_{\bar z}}\,|\,\mathbf{z}'\rangle\theta(t)\ , \tag{8.34}$$

which will be given by the perturbative expansion

$$e^{\Sigma(\mathbf{z})/2}\mathcal{K}^t_\Sigma(\mathbf{z},\mathbf{z}')e^{\Sigma(\mathbf{z}')/2}=\hat{\mathcal{K}}^t(\mathbf{z},\mathbf{z}')+\int_0^t dt_1\int d^2\mathbf{z}_1\hat{\mathcal{K}}^{t-t_1}(\mathbf{z},\mathbf{z}_1)W(\mathbf{z}_1)\hat{\mathcal{K}}^{t_1}(\mathbf{z},\mathbf{z}')+\cdots\ , \tag{8.35}$$

where

$$\begin{aligned}\hat{\mathcal{K}}^t(\mathbf{z},\mathbf{z}')&=\frac{1}{4\pi t}\exp\left[-\frac{|z-z'|^2}{2t}\right](\theta'-\theta)\ ,\\ W(\mathbf{z})&=(e^{-2\Sigma(\mathbf{z})}-1)\frac{\partial}{\partial t}+2(\partial_{\bar z}e^{-2\Sigma(\mathbf{z})})\partial_{\bar z}\ .\end{aligned} \tag{8.36}$$

The contractions can be obtained from

$$\langle\Psi\bar\partial_z\Psi\rangle=\partial^m_{z'}\left[e^{-\Sigma(\mathbf{z}')/2}\int d^2\mathbf{w}\,\mathcal{K}^\varepsilon_\Sigma(\mathbf{z},\mathbf{w})e^{\Sigma(\mathbf{w})/2}\hat{\mathcal{D}}_+^{-1}(\mathbf{w},\mathbf{z}')\right]\Big|_{\mathbf{z}=\mathbf{z}'}\ . \tag{8.37}$$

As an example we get

$$\langle\Psi\mathcal{D}_{\bar z}\Psi\rangle=\frac{i}{2}\mathcal{D}_{\bar z}{}^2\Sigma+\frac{i}{3}(\mathcal{D}_{\bar z}\Sigma)^2\ . \tag{8.38}$$

It is clear that all other contractions could be derived in an analogous fashion, albeit by rather lengthy calculations.

Vertices in the operator language for the heterotic string may be found in Gross *et al.* (1986). The above formulas for the type-II superstring and heterotic strings, as well as the evaluation of super Weyl anomalies, are in D'Hoker and Phong (1987b).

E. The covariant fermion emission vertex operator and space-time supersymmetry

1. Covariant fermion emission vertex operator

Two related problems arise in the construction of fermion emission vertex operators, which together point to a solution. The first is to manufacture space-time spinors out of the worldsheet fermions ψ^μ which transform rather as an SO(10) vector. The second is that inserting a fermion emission vertex operator at a point z on the worldsheet must introduce a branch cut originating at z. To see this, we recall that in canonical quantization, free strings propagate along cylinders, and fermions and bosons correspond, respectively, to states in which ψ^μ are periodic $\psi^\mu(\sigma+\pi)=\psi^\mu(\sigma)$ (Ramond sector), and states in which ψ^μ are antiperiodic $\psi^\mu(\sigma+\pi)=-\psi^\mu(\sigma)$ (Neveu-Schwarz sector). Thus a fermion emission vertex must switch boundary conditions in order to preserve spin statistics. The way to achieve this is to introduce a cut originating at the insertion that contributes a factor of -1 when we cross it (see Fig. 23). Now a cut must originate at one point and end at some other point on the worldsheet. This means that we should look for operators S such that correlations of two S's with the ψ^μ's will be defined on the double cover of the worldsheet with branch points at the S insertions. Such operators can be obtained by bosonizing the worldsheet fermions. We

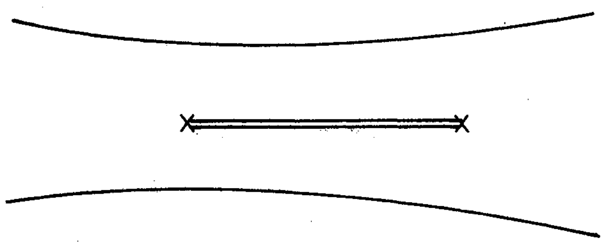

FIG. 23. Quadratic branch cut introduced by a fermion emission vertex.

shall see that they can also be combined into spinor representations of SO(10).

If we group the fields ψ^μ into

$$\psi^a_+=\psi^a+i\psi^{a+5}, \quad \psi^a_-=\psi^a-i\psi^{a+5},$$

they can be represented by scalar bosons ϕ^a,

$$\psi^a_\pm=e^{\pm i\phi^a}. \tag{8.39}$$

The spin fields S_α can next be defined by

$$\begin{aligned} &S_{\pm a}=e^{\pm i\phi^a/2}, \\ &S_\alpha=e^{\pm i\phi^1/2}\cdots e^{\pm i\phi^5/2}, \\ &\alpha=(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}). \end{aligned} \tag{8.40}$$

Now a key observation is that the integrals of the currents

$$J_a=-i\partial_z\phi_a \tag{8.41}$$

for the bosonized theories can be viewed as generating a Cartan subalgebra of SO(10), with the weights of S_α given by $\alpha=(\pm\frac{1}{2},\ldots,\pm\frac{1}{2})$. These are precisely the weights of the spinor representation of SO(10).

Since the ψ^μ's have conformal dimension $\frac{1}{2}$, it follows from Sec. II.J that the spin fields $S_{\pm a}$ have dimensions $\frac{1}{8}$, and hence the operators S_α have dimension $\frac{5}{8}$. Recall that a physical vertex operator must have conformal dimension 1, so that it may be integrated on the worldsheet. This appearance of the dimension $\frac{5}{8}$ instead of 1 was one of the major difficulties of fermion emission amplitudes already encountered in dual models, where a number of cures involving either projections onto physical states or a light-cone gauge approach were proposed. A natural resolution in the covariant formalism is in terms of the superconformal ghosts. Indeed, when a fermion vertex is inserted, it changes the boundary conditions of the worldsheet fermions and consequently the boundary conditions of the gravitino field χ, since χ couples to ψ^μ. In particular, the Grassmann parameter for supersymmetry transformations must be double valued around the insertions. In terms of the superconformal ghosts, this means that we must introduce a cut in the β,γ fields as well.

To achieve this, we need, as before, to bosonize the ghosts. There is an added complication here, since the superconformal ghosts are bosons, and $e^{\pm i\sigma}$ obey Fermi statistics. A way out is given by Friedan, Shenker, and Martinec (1985; Friedan, Martinec, and Shenker, 1986), who introduced fermion fields ξ and η, so that

$$\beta(z)=\partial_z\xi(z)e^{i\sigma(z)}, \quad \gamma(z)=\eta(z)e^{-i\sigma(z)} \tag{8.42}$$

are consistent with spin statistics. Spin fields $\Sigma_{\pm 1/2}$ in the fields of the superconformal ghosts can now be introduced by

$$\Sigma_{\pm 1/2}=e^{\pm i\sigma(z)/2}. \tag{8.43}$$

Since the β,γ have conformal dimensions $\frac{3}{2}$ and $-\frac{1}{2}$, the bosonized theory is coupled to a background charge $Q=-2$, the minus sign being due to the fact that the operator product expansion $\beta(z)\gamma(w)$ is now $-1/(z-w)$ instead of $1/(z-w)$. Similarly the conformal dimension of $e^{iq\sigma}$ in Eq. (2.184) becomes now $-q(q+Q)/2$, so that the conformal dimensions of $\Sigma_{+1/2}$ and $\Sigma_{-1/2}$ are $\frac{3}{8}$ and $-\frac{5}{8}$, respectively. This suggests that the fermion emission vertex is given by

$$V^F_{-1/2}(u^\alpha,k^\mu)=\Sigma_{1/2}(u^\alpha S_\alpha)e^{ik_\mu x^\mu}, \tag{8.44}$$

where $u^\alpha(k)$ is a space-time spinor. The correct conformal dimension is achieved for $k^\mu\Gamma_\mu u=k^2=0$, so that the space-time fermion emitted is massless. (Here Γ denote the ten-dimensional gamma matrices.)

It is useful to know the operator product expansions of the S_α's. In ten dimensions, the charge-conjugation matrix C is antisymmetric and interchanges chiralities. Left and right spinors correspond to an even or odd number of $+\frac{1}{2}$ in α. We should have then

$$\begin{aligned} S_\alpha(z)S^\beta(w)&\sim\frac{-\delta_\alpha{}^\beta}{(z-w)^{5/4}}+\frac{(\Gamma_\mu)_\alpha{}^\beta\psi^\mu}{(z-w)^{3/4}} \\ &\quad+\frac{(\Gamma_{\mu\nu})_\alpha{}^\beta\psi^\mu\psi^\nu}{(z-w)^{1/4}}+\cdots, \\ \psi^\mu(\zeta)S_\alpha(z)S_\beta(w)&\sim\frac{(\Gamma^\mu)_{\alpha\beta}}{(\zeta-z)^{3/4}}+\cdots, \\ J^{\mu\nu}(\zeta)S_\alpha(w)&\sim\frac{(\Gamma^{\mu\nu})_\alpha{}^\beta S_\beta(z)}{\zeta-w}+\cdots. \end{aligned} \tag{8.45}$$

To compute tree-level fermion scattering amplitudes for four or fewer fermions, we note that the amplitudes decompose into separate functions for the vertex operators e^{ikx}, for the bosonized superghosts, and for the spin operators. The first two types of correlation functions are familiar by now, and we can use Eq. (8.45) and ten-dimensional spinor algebra to arrive at projectively invariant expressions consistent with SO(10) transformation properties. The results reproduce those obtained by the earlier methods.

In general we also need another fermion emission vertex $V^F_{1/2}$ with opposite ghost charge. It is natural to make use of $\Sigma_{-1/2}$, which, combined with S_α, will however produce an operator of dimension 0. We have then to introduce a dimension-1 bosonic vertex. The vertex

$V_{1/2}^F$ is obtained by choosing the massless boson emission vertex

$$V_{1/2}^F = u^\alpha(k)\Sigma_{-1/2}(\partial x^\mu + k_\nu \psi^\nu \psi^\mu)\Gamma_\mu^{\alpha\beta} S_\beta e^{ikx} . \tag{8.46}$$

The vertices $V_{-1/2}^F$ and $V_{1/2}^F$ are just two of an infinite number of versions of the fermion emission vertex, which are actually related to one another by

$$\begin{aligned} V_{1/2}^F &= [Q_{\rm BRST}, \xi V_{-1/2}^F] , \\ V_{3/2}^F &= [Q_{\rm BRST}, \xi V_{1/2}^F] . \end{aligned} \tag{8.47}$$

The operators on the right-hand side are BRST invariant (as physical vertices must be), but not spurious despite the fact that they appear as BRST transforms. The reason is that the irreducible representation of the current algebra is built out of $\phi, \eta, \partial\xi$, but not of ξ itself. These many versions are caused by the necessity of prescribing an arbitrary Bose sea level for the superconformal ghosts. The Hilbert space of states must include all representations corresponding to various choices of sea levels. The vertex V_λ^F is the BRST-invariant vertex operator with Bose sea level λ. These ubiquitous "picture-changing" phenomena are explained in detail in Friedan, Martinec, and Shenker (1986). They are crucial in the construction of the ☆ operation of Witten's superstring field theory (1986b) and, as shown by Verlinde and Verlinde (1987b), in the gauge-fixed superstring of Sec. III.P.

Finally, we note that N-fermion emission for $N \geq 6$ is already more complicated even at tree level. In fact the above discussion shows that the worldsheet has to be viewed as a sphere with N punctures, and such a surface admits $(N-4)/2$ supermoduli parameters. Proper treatment of integration over moduli parameters presents some of the problems encountered earlier in multiloop amplitudes.

2. Space-time supersymmetry

The supersymmetry charge in the covariant formalism can now be obtained as a contour integral,

$$Q_\alpha = \oint \frac{dz}{2\pi i} V_\alpha^F(k=0) , \tag{8.48}$$

where $V_\alpha^F(k=0) = \Sigma_{1/2} S_\alpha$ is the fermion emission vertex at zero momentum. From operator product expansions we can check that Q_α transforms massless fermion vertices into massless boson vertices and vice versa,

$$\begin{aligned} \{Q_\alpha, V^F(u,k)\} &= V^B(\zeta^\mu = u\Gamma^\mu, k) , \\ [Q_\alpha, V^B(\zeta,k)] &= V^F(u = ik^\mu \Gamma_{\mu\nu}\zeta^\nu, k) . \end{aligned} \tag{8.49}$$

In fact, the full supersymmetry algebra

$$\begin{aligned} [P^\mu, P^\nu] &= [P^\mu, Q_\alpha] = 0 , \\ \{Q_\alpha, Q_\beta\} &= (\Gamma^\mu)_{\alpha\beta} P^\mu , \end{aligned} \tag{8.50}$$

is obtained by taking P^μ to be the contour integral of the vector emission vertex at zero momentum: ∂x^μ. Powerful nonrenormalization theorems can in principle be deduced from this setup. For example, since vertices for massless bosons and fermions can be obtained from one another by a contour integral of the supersymmetry current [see Eq. (8.49)], boson propagator corrections follow from fermion propagator corrections by deformation of contours. Thus there will be no mass renormalization for massless bosons if the fermions are chiral.

It should be noted that this discussion is only local. For worldsheets of nontrivial topology, the supersymmetry current develops unphysical poles, which must be taken into account before any firm conclusion can be drawn. Furthermore, there may be contributions from the boundary of moduli space.

Some classical papers on the fermion emission vertex are those of Schwarz and Wu (1971), Thorn (1971), Corrigan and Olive (1972), Brink *et al.* (1973), and Mandelstam (1974a), the last paper being based on path-integral methods in the light-cone gauge. The covariant fermion vertex operator was constructed by Friedan, Shenker, and Martinec (1985; Friedan, Martinec, and Shenker, 1986) and Knizhnik (1985, 1986a, 1986b). That the ghosts should contribute was suggested by Goddard and Olive. A derivation of quantum numbers for the vertex from the Polyakov integral on surfaces with punctures was proposed by Knizhnik (1986a, 1986b). Explicit calculations of fermion emissions in the covariant formalism are given in Cohn *et al.* (1986), Knizhnik (1986a, 1986b), and Kostelecky, Lechtenfeld, and Samuel (1987). Nonrenormalization theorems are in Martinec (1986). Picture-changing phenomena were uncovered by Friedan, Martinec, and Shenker (1986). Their role in superstring field theory is discussed by Witten (1986b), and in the gauge-fixed multiloop partition function by Friedan, Martinec, and Shenker (1986) and Verlinde and Verlinde (1987b), as we saw earlier in Sec. III.P. Verlinde and Verlinde discovered the unphysical poles in the supersymmetry current and argued that their residues must be total derivatives on moduli space. They also provided expressions for the correlation functions of the bosonized superconformal ghosts in terms of the prime form. Unphysical poles as well as contributions from the boundary of moduli space in the two-loop case are treated by Atick and Sen (1987a, 1987b, 1987c).

IX. CONCLUSION AND OUTLOOK

In this paper we have reviewed some of the most recent developments in string perturbation theory. We shall now give a brief survey of the main objectives achieved so far, as well as of the questions that remain. We shall also take the opportunity to mention developments in other directions and include some references that have not occurred earlier in the text.

The structure of strings is amazingly rich, and in many ways quite rigid. Progress in the study of the bosonic string has been spectacular thanks to the concerted efforts of many authors, and we have now a very good understanding of scattering amplitudes, order by order in perturbation theory. The fermionic strings, on the other hand, have revealed themselves to be much more profound and fraught with dangerous subtleties. Their investigation has forced us to come to grips with some of the deepest questions in geometry. Nevertheless we have reached a stage now where the required machinery is in place, and the proposals we described in Secs. III.K, III.O, and VII.G point to a consistent formulation of superstrings. More specifically, with internal loop momenta, we have a way of separating left from right chiralities, which does reproduce the heterotic string from the chirally split RNS string. This way is also precisely the one agreeing with holomorphic splitting on supermoduli space, allowing us to integrate out the odd moduli, and holomorphic splitting on supermoduli will then reduce to holomorphic splitting on "moduli," if "moduli" space is viewed as the $(3h-3)$-dimensional space of supersymmetry period matrices $\hat{\Omega}$. The formulation can then be argued to lead to modular-invariant amplitudes, even taking into account the fact that no global section over moduli space of $\frac{3}{2}$ differentials can be chosen to gauge-fix the superstring. It also offers a way out of the apparent ambiguities of the picture-changing formalism discussed by Atick, Rabin, and Sen (1987), Moore and Morozov (1987) and Verlinde (1987). These ambiguities, for example, could have led to a nonvanishing cosmological constant at two-loop order if not treated properly.

A more explicit implementation with the required technology of the above program is the natural next step. Here we are encouraged by the rapid progress in the understanding of two-dimensional supergeometry and supermoduli space. Difficulties with indefinite metrics have been resolved (Secs. III.F and III.H), a complex structure of supermoduli space has been introduced (Secs. III.G and VII.F), and foundations of superalgebraic geometry are on the way with the super Abelian differentials and supersymmetric period matrix (Secs. VII.F and VII.G; Sonoda, 1987b). Line bundles over super Riemann surfaces have been investigated by Giddings and Nelson (1987). It is perhaps timely to formulate and solve a Schottky problem for supersymmetric period matrices. From the component point of view, we now have at our disposal the chiral bosonization formulas for ghosts and superghosts of Verlinde and Verlinde (1987a, 1987b), as well as a good understanding of Mandelstam diagrams (Sec. IV.G; Giddings and Wolpert, 1987) and of relations between their determinants (Sec. V.G). That the present formulation is an efficient tool for practical calculations is illustrated to one loop in Sec. III.M. All this is grounds for believing that we shall shortly have explicit confirmation of consistency and unitarity of superstrings, together with simple rules for calculating scattering amplitudes.

In this paper we have discussed only briefly the picture-changing formalism and the necessary Wu-Yang correction terms, and we have not pursued it further. This is clearly an important issue, since it is intimately connected with manifest BRST invariance. A detailed discussion of this topic and of whether supermoduli space splits over moduli is to be found in Verlinde (1987). Other options have been suggested by Atick, Moore, and Sen (1988a, 1988b).

Perhaps after mastering the subtleties of string perturbation theory we may find a mechanism for breaking supersymmetry while maintaining a vanishing cosmological constant. A proposal based on modular forms to one loop has been presented by Moore (1987).

In a different direction, the string ground state should be determined by physics at the Planck scale, and formally perturbative amplitudes may be used to probe the higher-energy (limit as the Planck mass tends to zero) behavior of string theory. Such investigations have been initiated by Gross and Mende (1987, 1988) and Gross (1988), who argue that in the $T\rightarrow 0$ limit, contributions from surfaces with discrete symmetry dominate, and an infinite number of relations then hold between scattering amplitudes. This suggests the presence of a huge spontaneously broken symmetry.

At the other end, in the low-energy limit ($T\rightarrow\infty$), string theory should reduce to an effective field theory, whose equations of motion are given by the requirement of conformal invariance. Thus a vacuum configuration corresponds to a conformal field theory. Of particular interest are vacuum configurations in which space-time splits into four-dimensional Minkowski space-time times a six-dimensional internal space M_6. Vanishing of the beta functions as well as unbroken $N=1$ supersymmetry restricts M_6 to be essentially a Calabi-Yau (i.e., Ricci flat and Kähler) manifold. This was argued by Callan, Friedan, Martinec, and Perry (1985), Candelas, Horowitz, Strominger, and Witten (1985), Green, Schwarz, and West (1985), Sen (1985, 1986a), Grisaru, Van de Ven, and Zanon (1986), Howe, Papadopoulos, and Stelle (1986), and Witten (1986a, 1986b). Other conformal field theories are provided by orbifolds, introduced by Dixon, Harvey, Vafa, and Witten (1985, 1986), toroidal compactifications (Narain, 1986; Ginsparg and Vafa, 1987; Narain and Sarmidi, 1987; Narain, Sarmidi, and Vafa, 1987; Narain, Sarmidi, and Witten, 1987), quasicrystalline orbifolds (Harvey, Moore, and Vafa, 1988), and group manifolds (Jain, Shankar, and Wadia, 1985; Gepner and Witten, 1986; Jain, Mandal, and Wadia, 1987). The moduli space of conformal field theories and renormalization-group equations are considered, respectively, in Seiberg (1987) and Banks and Martinec (1987). Four-dimensional theories from the $d=10$ type-II theories with chiral asymmetry are constructed by Antoniadis *et al.* (1986), Bluhm, Dolan, and Goddard (1987), Dixon, Kaplunovsky, and Vafa (1987), and Kawai, Lewellen, and Tye (1987).

The large number of candidate vacua will require a better understanding of nonperturbative effects, for example, of stringy instantons. Very early on in string theory, attempts were made to derive string perturbation theory from a string field theory, in the hope that string field theory might be consistently interpolated off-shell. Some of the earliest works are those of Mandelstam (1973a, 1973b), Cremmer and Gervais (1974), and Kaku and Kikkawa (1974). More recently, superstring fields in the light-cone gauge have been formulated by Green and Schwarz (1983, 1984), Green, Schwarz, and Brink (1983), and Gross and Periwal (1988), although in a background-dependent way. Covariant formulations requiring an unphysical length parameter are presented by Kazama *et al.* (1986), Hata *et al.* (1987), and Neveu and West (1987). String fields based on BRST invariance have been developed by Friedan (1985), Siegel (1985), Siegel and Zwiebach (1986), Banks and Peskin (1986), and Witten (1986a, 1986b). Witten's theory is based on a remarkable interaction on the worldsheet. Its bosonic version has been shown to reproduce the correct (open-string) amplitudes by Giddings (1986), Giddings and Martinec (1986), Giddings, Martinec, and Witten (1986), and Thorn (1987). Background-independent formulations for it have been proposed by Horowitz *et al.* (1986), as well as closed-string versions by Strominger (1987). Operator formulations have been worked out by Gross and Jevicki (1987).

More radical proposals for the study of nonperturbative effects have been put forth by Friedan and Shenker (1986, 1987) and by Bowick and Rajeev (1987). Friedan and Shenker use factorization requirements to lump moduli spaces of all genera, including surfaces with nodes, into a universal moduli space. An abstract string theory corresponds to a holomorphic vector bundle together with a flat connection on the universal moduli space. Nonperturbative effects correspond to a completion of the universal moduli space, which must then include some classes of surfaces of infinite genus. This approach has been extended to the case of the superstring by Cohn (1988). On the other hand, Bowick and Rajeev (1987) view conformal invariance as invariance under $\mathrm{Diff}S^1/S^1$, so that the key requirement becomes flatness and trivial holonomy of parallel transport along $\mathrm{Diff}S^1/S^1$. Now $\mathrm{Diff}S^1/S^1$ is a Kähler manifold with nonvanishing Ricci curvature, and an acceptable theory can be viewed as a vector bundle on this space, whose curvature cancels that of the tangent bundle. This would be an analog of the anomaly cancellation between matter and ghost parts in the Polyakov string. Related ideas have been developed by Witten (1987).

A natural setup in which Riemann surfaces (more precisely, with a puncture and a local coordinate system) of all genera appear on the same footing is provided by Sato's universal Grassmannian, which has been at the center of great developments in connection with integrable systems. It is similar in many ways to moduli space and is already known to provide an operator proof of Bose-Fermi equivalence (Sato, Jimbo, and Miwa, 1977, 1978, 1979; Date *et al.* 1983; Segal and Wilson, 1985). Possibilities of string theory formulations in terms of Grassmannians are investigated by Ishibashi, Matsuo, and Ooguri (1986), Alvarez-Gaumé, Gomez, and Reina (1987), Vafa (1987), and Witten (1988a, 1988b). Grassmannians and the homology of the mapping class group are studied by Arbarello *et al.* (1987). Of course, for the fermionic string we would need an analog of this theory based on super Riemann surfaces.

Finally, several authors have also suggested considering string theories over number fields different from the complex. This may help to solve some theories, as well as to uncover any arithmetic structure that may make the theory even more rigid.

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APPENDIX A: CONVENTIONS

1. Differential geometry

We use the following conventions for covariant derivatives and connections:

$$\begin{aligned} \nabla_m V_n &= \partial_m V_n - \Gamma^p_{mn} V_p \ , \\ \nabla_m V^n &= \partial_m V^n + \Gamma^n_{mp} V^p \ . \end{aligned} \tag{A1}$$

Generalizations to tensors of arbitrary rank may be deduced by applying covariant derivatives to tensor products and using Leibnitz's rule.

The Riemann curvature tensor is given by

$$R^l{}_{mnk}=\frac{\partial\Gamma^l_{mn}}{\partial x^k}-\frac{\partial\Gamma^l_{mk}}{\partial x^n}+\Gamma^p_{mn}\Gamma^l_{kp}-\Gamma^p_{mk}\Gamma^l_{np} \tag{A2}$$

or

$$[\nabla_m,\nabla_n]V_k=-R^l{}_{kmn}V_l \,. \tag{A3}$$

One has the symmetry properties

$$\begin{aligned}&R_{lmnk}=R_{nklm}\,,\\ &R_{lmnk}=R_{mlkn}=-R_{mlnk}=-R_{lmkn}\,,\\ &R_{lmnk}+R_{lkmn}+R_{lnkm}=0\,.\end{aligned} \tag{A4}$$

In Riemannian geometry, the connection Γ^p_{mn} is symmetric (i.e., has zero torsion) and the metric is covariantly constant,

$$\nabla_k g_{mn}=0\,.$$

Hence the Levi-Civitá connection is given by

$$\Gamma^p_{mn}=\tfrac{1}{2}g^{pq}(\partial_m g_{nq}+\partial_n g_{mq}-\partial_q g_{mn})\,. \tag{A5}$$

One also defines the Ricci curvature tensor

$$R_{mn}=R^l{}_{mln}$$

and the (Gaussian) scalar curvature

$$R=-\tfrac{1}{2}g^{mn}R_{mn} \quad\text{or}\quad R_{mn}=-g_{mn}R\,. \tag{A6}$$

A change in the Levi-Civitá connection is a tensor, given by

$$\delta\Gamma^p_{mn}=\tfrac{1}{2}g^{pq}(\nabla_m\delta g_{nq}+\nabla_n\delta g_{mq}-\nabla_q\delta g_{mn})\,, \tag{A7}$$

generating a change in the Riemann curvature,

$$\delta R^l{}_{mnk}=\nabla_k\delta\Gamma^l_{mn}-\nabla_n\delta\Gamma^l_{mk}\,,$$

and in the Gaussian curvature,

$$\begin{aligned}\delta R=&-\tfrac{1}{2}Rg^{mn}\delta g_{mn}-\tfrac{1}{2}\nabla_p\nabla^p(g^{mn}\delta g_{mn})\\ &+\tfrac{1}{2}\nabla^m\nabla^n\delta g_{mn}\,,\end{aligned} \tag{A8}$$

which is Eq. (2.34). It is also useful to record the changes in the covariant derivatives,

$$\delta\nabla^z=-\delta\sigma\nabla^z+\tfrac{1}{2}\delta g^{zz}\nabla_z+\frac{n}{2}\nabla_z(\delta g^{zz})\,,$$

$$\delta\nabla_z=-n\,\partial_z\delta\sigma-\tfrac{1}{2}\delta g_{zz}\nabla^z+\frac{n}{2}\nabla^z(\delta g_{zz})$$

with $\delta g^{zz}=-(g^{z\bar z})^2\delta g_{\bar z\bar z}$.

General two-dimensional coordinates are denoted by ξ^1 and ξ^2 or $\xi=(1/\sqrt{2})(\xi^1+i\xi^2)$. The metric is then $ds^2=g_{mn}d\xi^m d\xi^n$. Locally conformally flat coordinates are denoted by z (or w, etc.) and the metric is then $ds^2=2g_{z\bar z}dz\,d\bar z$. With these conventions, one has the following metrics and curvatures, $g_{z\bar z}R=-\partial_z\partial_{\bar z}\ln g_{z\bar z}$, for the sphere, plane, and upper half plane, respectively:

$$\begin{aligned}&ds^2=\frac{4\,dz\,d\bar z}{(1+|z|^2)^2}\,,\quad R=1\,,\\ &ds^2=2\,dz\,d\bar z\,,\quad R=0\,,\\ &ds^2=-\frac{dz\,d\bar z}{(z-\bar z)^2}\,,\quad R=-1\,.\end{aligned}$$

The sphere has area 4π. Furthermore, the covariant derivatives in locally conformally flat coordinates are

$$\nabla_z V_{\underbrace{zz\cdots z}_{n}}=(\partial_z-n\Gamma^z_{zz})V_{\underbrace{zz\cdots z}_{n}}\,, \tag{A9}$$

encountered in Eqs. (2.41)–(2.44), with $\Gamma^z_{zz}=\partial_z\ln g_{z\bar z}$. We use the notation for flat metric $g_{z\bar z}=g_{\bar z z}=\sqrt{g}=1$:

$$d^2z\sqrt{g}=dx\,dy=i\,dz\wedge d\bar z\,. \tag{A10}$$

2. Spinors, Dirac matrices

U(1) vector indices are denoted by $a,b,\ldots$ and take on the values z and $\bar z$; spinor indices are denoted by $\alpha,\beta,\ldots$ and take values $+$ and $-$. We use the same notations z and $\bar z$ for conformally flat coordinates and U(1) indices because in conformally flat coordinate systems they may be identified, so no confusion should arise. Dirac matrices satisfy

$$\{\gamma^a,\gamma^b\}=-\delta^{ab},\quad [\gamma^a,\gamma^b]=-\varepsilon^{ab}\gamma_5\,. \tag{A11}$$

We take the convenient representation of this Clifford algebra,

$$\begin{aligned}&(\gamma^z)_{++}=(\gamma^{\bar z})_{--}=-(\gamma^z)_+{}^-=(\gamma^{\bar z})_-{}^+=1\,,\\ &(\gamma^a)_{\alpha\beta}=0,\quad \alpha\neq\beta\,,\\ &(\gamma_5)_{+-}=(\gamma_5)_{-+}=(\gamma_5)_+{}^+=-(\gamma_5)_-{}^-=i\,,\\ &(\gamma_5)_{\pm\pm}=0\,.\end{aligned} \tag{A12}$$

It is also useful to have the following formulas at hand:

$$\psi_+=-\psi^-,\quad \psi_-=\psi^+\,,$$

and the following conventions for the antisymmetric tensor:

$$\varepsilon^{\bar z z}=-\varepsilon^{z\bar z}=\varepsilon_{z\bar z}=-\varepsilon_{\bar z z}=i,\quad \varepsilon_{zz}=\varepsilon_{\bar z\bar z}=0\,. \tag{A13}$$

Contractions without indices written explicitly are understood as

$$\omega\psi=\omega^\alpha\psi_\alpha=-\omega_\alpha\psi^\alpha=\omega^+\psi_++\omega^-\psi_-\,. \tag{A14}$$

It is useful to have

$$\theta_\alpha\theta^\beta=\delta_\alpha{}^\beta\theta\bar\theta\,.$$

3. Covariant derivatives on U(1) tensors

For Weyl spinors of U(1) weight $\pm\frac{1}{2}$ we have

$$D_a\psi_\pm=e_a{}^m\left[\partial_m\pm\frac{i}{2}\omega_m\right]\psi_\pm\,.$$

On a Dirac spinor we have

$$D_a\psi_\alpha = e_a{}^m[\partial_m\psi_\alpha + \tfrac{1}{2}(\gamma_5)_\alpha{}^\beta\omega_m\psi_\beta] \ .$$

On a general tensor-spinor of weight n, we have

$$D_a\psi_{(n)} = e_a{}^m(\partial_m\psi_{(n)} + in\,\omega_m\psi_{(n)}) \ .$$

Here the spin connection is given by

$$\omega_m = -e_m{}^a\varepsilon^{pq}\partial_p e_q{}^b\delta_{ab} \ ,$$

where

$$\varepsilon^{pq} = e_a{}^p e_b{}^q\varepsilon^{ab} \ .$$

The Gaussian curvature is expressed in terms of the spin connection

$$R = \varepsilon^{mn}\partial_m\omega_n \ ,$$

which is Eq. (3.37).

4. Dirac singularity

By definition

$$\int d^2w\,\delta(z-w)f(w) = f(z) \ .$$

We shall also use the covariant Dirac delta function

$$\delta(z,w) = \frac{1}{\sqrt{g}}\delta(z-w) \ .$$

Notice the minus sign versus the comma. It may be viewed as the limit in the sense of distributions of certain functions as $\varepsilon\to 0$,

$$\frac{1}{2\pi}\frac{\varepsilon^2}{(|z-w|^2+\varepsilon^2)^2}\to\delta(z-w)$$

and

$$\frac{1}{4\pi\varepsilon}\exp\left[-\frac{1}{2\varepsilon}|z-w|^2\right]\to\delta(z-w) \ .$$

In particular, we have

$$\partial_{\bar z}\frac{1}{z-w} = 2\pi\delta(z-w) \ .$$

Notice the unusual factor of 2 in this convention.

APPENDIX B: SHORT-TIME EXPANSIONS OF THE HEAT KERNEL

The heat kernel for the operator $\Delta_n^{(-)}$ satisfies the equation

$$\left[\frac{\partial}{\partial t} + \Delta_n^{(-)}\right]K_n^t(z,z') = \delta^2(z,z')\delta(t) \qquad \text{(B1)}$$

with the solution

$$K_n^t = \theta(t)e^{-t\Delta_n^{(-)}} \ .$$

We wish to calculate elements on or close to the diagonal $z=z'$ for short times t. As the issues involved are exclusively local in z, we may locally perform a reparametrization rendering the metric conformal to the Euclidean metric: $g_{mn}(z) = e^{2\sigma(z)}\delta_{mn}$. From Eqs. (2.42)–(2.44) and (2.47), we readily find the σ dependence of $\Delta_n^{(-)}$:

$$\Delta_n^{(-)} = e^{-2\sigma}\Delta + 4n\,\partial_z\sigma e^{-2\sigma}\partial_{\bar z} \ , \qquad \text{(B2)}$$

where Δ is the flat-space Laplacian $\Delta = -2\partial_z\partial_{\bar z}$. Combining Eqs. (B1) and (B2) and the scaling of $\delta^2(z,z')$ under constant Weyl transformations σ_0, we find that

$$K_n^t(z,z') = e^{-2\sigma_0}K_n^{\hat t}(z,z') \quad \text{with } \hat t = e^{-2\sigma_0}t \ . \qquad \text{(B3)}$$

Thus, without loss of generality, we may assume that $\sigma=0$ at the point of interest z'. We now rewrite (B1) as

$$\left[\frac{\partial}{\partial t} + \Delta - V_n\right]K_n^t(z,z') = \delta^2(z,z')\delta(t) \qquad \text{(B4)}$$

with

$$V_n = (1-e^{-2\sigma})\Delta - 4n\,\partial_z\sigma e^{-2\sigma}\partial_{\bar z} \ .$$

The flat-space heat kernel satisfies the equation

$$\left[\frac{\partial}{\partial t} + \Delta\right]K^t(z,z') = \delta^2(z,z')\delta(t)$$

with explicit solution

$$K^t(z,z') = \frac{1}{4\pi t}e^{-|z-z'|^2/2t}\theta(t) \ . \qquad \text{(B5)}$$

With the flat-space heat kernel, we can derive an integral equation for K_n^t,

$$K_n^t = K^t + \int dt' K^{t-t'}V_n K_n^{t'} \ ,$$

where pairwise integrations over z coordinates are understood, and which is solved by the following formal infinite series:

$$K_n^t = K^t + \int dt' K^{t-t'}V_n K^{t'} + \int dt'\int dt'' K^{t-t'}V_n K^{t'-t''}V_n K^{t''} + \cdots \ . \qquad \text{(B6)}$$

1. The diagonal of the heat kernel

On the diagonal $z=z'$, K^t is of order $1/t$. V_n must involve at least one derivative on σ, and two-dimensional rotational invariance requires equal numbers of z and $\bar z$ derivatives, so that Eq. (B6) is easily seen to be an expansion in increasing powers of t, starting with $1/t$. Thus, in the short-time limit, we shall be interested in contributions with 0 derivatives coming from the first term (of order $1/t$) and with one z and one $\bar z$ derivative coming from the second term (of order t^0). Actually, the terms proportional to $\partial_z\sigma\partial_{\bar z}\sigma$ cancel between the second and third terms, as can be seen by a simple calculation not reproduced here. It remains to obtain the terms in $\partial_z\partial_{\bar z}\sigma$, which arise solely from the second term in (B6):

$$K_n^t(z,z)=\frac{1}{4\pi t}+\int_0^t dt'\int d^2z' K^{t-t'}(z,z')\bar{V}_n(z') \times K^{t'}(z',z)+O(t) , \quad (B7)$$

where the contribution of V_n proportional to $\partial_z\partial_{\bar z}\sigma$ is denoted by $\bar{V}_n$ and is given by

$$\bar{V}_n(z')=2\,|z'-z|^2(\partial_z\partial_{\bar z}\sigma)\Delta_{z'} -4n(\bar z'-\bar z)(\partial_z\partial_{\bar z}\sigma)\partial_{\bar z'} . \quad (B8)$$

It is straightforward to evaluate the necessary z integrals:

$$\int d^2z' K^{t-t'}(z,z')\,|z'-z|^2\Delta_{z'}K^{t'}(z',z) =\frac{1}{2\pi t^3}(t-t')(2t'-t)$$

and

$$\int d^2z' K^{t-t'}(z,z')(\bar z-\bar z')\partial_{\bar z'}K^{t'}(z',z)=\frac{1}{4\pi t^3}(t-t') .$$

Putting all together, one finds

$$K_n^t(z,z)=\frac{1}{4\pi t}+\frac{1-3n}{12\pi}\Delta\sigma , \quad (B9)$$

and taking into account the additional constant Weyl rescalings as given in Eq. (B3) one finds with the help of Eq. (2.31) that

$$K_n^t(z,z)=\frac{1}{4\pi t}+\frac{1-3n}{12\pi}R , \quad (B10)$$

whence Eq. (2.68).

2. The anomaly in the ghost number current

The ghost number current $j_z=c^z b_{zz}$ is naively analytic, but suffers an anomaly, which we shall now calculate. Observe that the ghost propagator

$$G(z,w)=\langle c^z b_{ww}\rangle \quad (B11)$$

is Weyl invariant off the diagonal. The regularized ghost number current may be defined in a reparametrization-invariant way with the help of the heat kernel and a short-time cutoff:

$$\langle j_z\rangle=\int d^2w\sqrt{g(w)}G(z,w)K_2^{\varepsilon}(w,z) , \quad (B12)$$

where K_2^t is the heat kernel defined above. The anomaly in $\nabla^z j_z$ is a local scalar function of dimension 2, dependent only on the metric and its derivatives. Thus it must be proportional to the curvature. The coefficient may be gotten by calculating the term proportional to $\partial_z\sigma$ is $\langle j_z\rangle$ in the limit where $\varepsilon\to 0$, and then taking the $\partial_{\bar z}$ derivative. To calculate $\langle j_z\rangle$ we use again the expansion (B6), but this time away from the diagonal. As one is interested in a contribution linear in σ, one need only retain the first two terms in Eq. (B6), and the relevant part of V_2 of Eq. (B4) is $\tilde{V}_2$:

$$\langle j_z\rangle=\int d^2w\, e^{2\sigma(w)}\frac{1}{z-w}\left[K^t(w,z)+\int_0^t dt'\int d^2z' e^{2\sigma(z')}K^{t-t'}(w,z')\tilde{V}_2(z')K^{t'}(z',z)\right] , \quad (B13)$$

where

$$\tilde{V}_2(z')=2(\partial_z\sigma)(z'-z)\Delta'-4n\,\partial_z\sigma\,\partial_{\bar z'} .$$

To leading order in t, one finds after some calculation

$$\partial_{\bar z}\langle j_z\rangle=-\tfrac{3}{2}R . \quad (B14)$$

3. Anomalous contractions

Since Weyl anomalies are purely local, we may work in local isothermal coordinates. The propagator may be regularized at short distances by convolution with the heat kernel, evaluated after a short time ε. The propagator at distinct points is Weyl invariant and given by

$$\Delta^{-1}(z,z')=-\frac{1}{4\pi}\ln|z-z'|^2, \quad z\neq z' . \quad (B15)$$

The regularized propagator instead is given by

$$\Delta_t^{-1}(z,z')=\int d^2w\,\Delta^{-1}(z,w)K_0^t(w,z') , \quad (B16)$$

where K_0^t may be thought of as given by the expansion (B6), but this time evaluated at distinct points w and z'. Contractions will be performed at some fixed point, say $z=0$. Fixing the overall scale so that $\sigma(0)=0$, we may expand the potential W about 0 to yield

$$W=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{1}{m!(m-k)!}(\partial^m\bar\partial^{k-m}e^{-2\sigma})(-\Delta)\Big|_{z=0} \times z^m\bar z^{k-m} . \quad (B17)$$

It is now simple algebra to substitute this into Eq. (B6) and evaluate perturbatively. Note that we need only deal with derivatives of the propagator, which simplifies matters. Thus we have

$$\langle x(0)\partial^m x(0)\rangle=4\pi\int d^2z\,K_0^t(0,z;\varepsilon)\partial_{z'}^m\Delta^{-1}(z,z')\Big|_{z'=0} =\int d^2z\,K_0^t(0,z;\varepsilon)(m-1)!z^{-m} . \quad (B18)$$

If $\langle x(0)\partial^m x(0)\rangle_p$ denotes the contribution to the above expression of the terms in (B6) with p interaction factors $W_{z_1},\ldots,W_{z_p}$, each given by a formula such as (B17) with exponents k_i and m_i, we must have from invariance under $z\to e^{i\theta}z$ and $z_i\to e^{i\theta}z_i$,

$$\sum_{i=1}^{p}(2m_i-k_i)=m .$$

On the other hand, simple power counting shows that

$$\sum_{i=1}^{p} k_i \le m \ ,$$

otherwise the limit as $\varepsilon \to 0$ will vanish. Thus $k_i = m_i$ for all i, and $\sum_{i=1}^{p} m_i = m$. This means that only ∂ derivatives on σ will appear in the final answer, and no $\bar\partial$. Furthermore, the interaction $W_{z_i} = (1-e^{-2\sigma})\Delta_{z_i}$ applied to $K^{t_i - t_{i+1}}(z_i, z_{i+1})$ produces a term

$$(e^{-2\sigma}-1)\frac{\partial}{\partial \tau_i} K^{\tau_i}(z_i, z_{i+1}) \ ,$$

where one sets $\tau_i = t_i - t_{i+1}$ after differentiation has been performed. Substituting in (B6) and carrying out the Gaussian integrals yields Eq. (8.11).

APPENDIX C: THE DIAGONAL OF THE SUPER HEAT KERNEL

We shall now compute the heat kernel for the superspace Laplacian $(\Box_n^-)^2$, satisfying the equation

$$\left[\frac{\partial}{\partial t} + (\Box_n^-)^2\right] \mathcal{K}_n^t(z,z';\theta,\theta') = \delta^2(z,z')\delta^2(\theta,\theta')\delta(t) \tag{C1}$$

with solution

$$\mathcal{K}_n^t = \theta(t) e^{-t(\Box_n^-)^2} \ .$$

We are specially interested in calculating elements on the diagonal: $z=z'$, $\theta=\theta'$ for short times t. This problem is entirely local, so we may perform a super-reparametrization and local U(1) transformation to render the geometry superconformally flat. From Eq. (3.101) we find the super Weyl dependence of $(\Box_n^-)^2$,

$$\begin{aligned}\Box_n^- V &= \mathcal{D}_+^{n-1/2}\mathcal{D}_-^n V \\ &= e^{(n-1)\Sigma}\hat{\mathcal{D}}_+^{n-1/2} e^{-2n\Sigma}\hat{\mathcal{D}}_-^n e^{n\Sigma} V \ .\end{aligned} \tag{C2}$$

Since the quantities with hats are taken with respect to flat supergeometry, we may replace $\hat{\mathcal{D}}_\pm^n = \partial_\pm$, so that

$$\begin{aligned}\Box_n^- V &= e^{(n-1)\Sigma}\partial_+[e^{-n\Sigma}(\partial_- V + n\partial_-\Sigma V)] \\ &= e^{-\Sigma}(\partial_+\partial_- V + n\partial_+\partial_-\Sigma V - n\partial_-\Sigma\partial_+ V \\ &\quad - n\partial_+\Sigma\partial_- V - n^2\partial_+\Sigma\partial_-\Sigma V)\end{aligned}$$

and its square is a very lengthy expression which can be worked out in a straightforward manner. Actually, we shall be interested only in the contribution that has at most two superderivatives on Σ fields in total, the other contributions tending to 0 at $t \to 0$. The two derivatives, moreover, must be one ∂_+ and one ∂_- in order to get a nonzero answer. With this restriction, we get

$$\begin{aligned}(\Box_n^-)^2|_{\text{restr.}} V = e^{-2\Sigma}[&\partial_+\partial_-\partial_+\partial_- V - n\partial_-\Sigma\partial_+\partial_-\partial_+ V - (2n-1)\partial_-\partial_+\Sigma\partial_+\partial_- V + \partial_+\Sigma\partial_+\partial_-^2 V \\ &- n^2\partial_+\Sigma\partial_-\Sigma\partial_+\partial_- V - (n-1)\partial_-\Sigma\partial_+^2\partial_- V + n(n-1)\partial_-\Sigma\partial_+\Sigma\partial_+\partial_- V \\ &- n(n+1)\partial_+\Sigma\partial_-\Sigma\partial_-\partial_+ V - (n^2-1)\partial_+\Sigma\partial_-\Sigma\partial_+\partial_- V] \ .\end{aligned} \tag{C3}$$

The contributions from the terms quadratic in Σ are easily seen to vanish in computing the kernel at coincident points, so that we are effectively left with

$$\begin{aligned}(\Box_n^-)^2 V = e^{-2\Sigma}[&-\partial_+^2\partial_-^2 V + \partial_-\Sigma\partial_-\partial_+^2 V + \partial_+\Sigma\partial_+\partial_-^2 V \\ &+ (2n-1)\partial_+\partial_-\Sigma\partial_+\partial_- V] \ .\end{aligned} \tag{C4}$$

As in the bosonic case, we may omit the scalings by constant Σ and easily restore them at the end. Thus, without loss of generality, we may assume that $\Sigma=0$ at the point of interest (z',θ'). Using the fact that $-\partial_+^2\partial_-^2 = \frac{1}{2}\Delta$, where the flat-space Laplacian is given by $\Delta = -2\partial_z\partial_{\bar z}$, we rewrite the equation defining the heat kernel as

$$\left[\frac{\partial}{\partial t} + \Delta - \mathcal{V}_n\right]\mathcal{K}_n^t(z,z';\theta,\theta') = \delta^2(z,z')\delta^2(\theta,\theta')\delta(t) \ . \tag{C5}$$

The part $\overline{\mathcal{V}}_n$ relevant to the diagonal of $\mathcal{K}_n^t$ in $\mathcal{V}_n$ is given by

$$\begin{aligned}\overline{\mathcal{V}}_n = (1-e^{-2\Sigma})\Delta - e^{-2\Sigma}[&\partial_-\Sigma\partial_+^2\partial_- - \partial_+\Sigma\partial_+\partial_-^2 \\ &-(2n-1)\partial_+\partial_-\Sigma\partial_+\partial_-] \ .\end{aligned} \tag{C6}$$

The flat-space heat kernel satisfies

$$\left[\frac{\partial}{\partial t} + \Delta\right]\mathcal{K}^t(z,z';\theta,\theta') = \delta^2(z,z')\delta^2(\theta,\theta')\delta(t) \ ,$$

which is solved by

$$\mathcal{K}^t(z,z';\theta,\theta') = K^t(z,z')\delta^2(\theta,\theta')$$

where the usual flat-space heat kernel K^t is given by Eq. (B5). The major distinction from the bosonic case is that $\mathcal{K}^t$ vanishes on the diagonal because $\delta^2(\theta,\theta)=0$, which implies that there is *no* term in t^{-1} in the expansion of $\mathcal{K}_n^t$ in terms of small t. In analogy with Eqs. (B6) and (B7) we readily find that the relevant contributions are

$$\begin{aligned}\mathcal{K}_n^t(z,z;\theta,\theta) = \int_0^t dt' \int d^2z' d^2\theta' \mathcal{K}^{t-t'}(z,z';\theta,\theta') \\ \times \overline{\mathcal{V}}_n(z',\theta')\mathcal{K}^{t'}(z',z;\theta',\theta) \\ + O(t) \ .\end{aligned}$$

It is convenient first to work out the θ' integral:

$$\begin{aligned}\int d^2\theta' \delta^2(\theta,\theta')\overline{\mathcal{V}}_n(z',\theta')\delta^2(\theta',\theta) &= \overline{\mathcal{V}}_n(z',\theta')\delta^2(\theta',\theta)|_{\theta'=\theta} \\ &= -(2n-1)\partial_+\partial_-\Sigma\end{aligned} \tag{C7}$$

so that

$$\mathcal{K}_n^t(z,z;\theta,\theta) = -(2n-1)\partial_+\partial_-\Sigma \times \int_0^t dt' \int d^2z' K^{t-t'}(z,z')K^{t'}(z',z) = -\frac{2n-1}{2\pi}\partial_+\partial_-\Sigma \ . \tag{C8}$$

Restoring the factor of constant Σ scalings and using Eq. (3.20) for R_{+-}, we find

$$\mathcal{K}_n^t(z,z;\theta,\theta) = -i\frac{2n-1}{4\pi}R_{+-} + O(t) \ . \tag{C9}$$

An analogous calculation for $\Box_n^+$ will give a coefficient $2n+1$ in front of R_{+-} instead of $2n-1$.

APPENDIX D: RIEMANN VANISHING AND ABEL THEOREMS

In this appendix we shall present some of the methods of the theory of Riemann surfaces, and in particular provide proofs for some of the properties of the period matrix and theta functions used in the text. The key tool is Green's theorem on a cut Riemann surface. Recall that we can choose a homology basis satisfying the intersection pairings (3.5), and that representatives of the cycles A_I, B_I, $I=1,\ldots,h$ in the basis may be chosen as in Fig. 10. It is not difficult to see that the surface M can then be cut along these cycles in a $4h$ polygonal region (see Fig. 11).

Here we have labeled by $+$ and $-$ the oriented edges of each cycle, and the oriented boundary of the cut Riemann surface M_{cut} is

$$\partial M_{\text{cut}} = -\sum A_I^+ - \sum B_I^+ + \sum A_I^- + \sum B_I^- \ .$$

The advantage of working with a cut surface is that any holomorphic differential ω can be integrated, $\omega = dg$, where g is a holomorphic function on M_{cut} with, however, different values on the $+,-$ edges of each cycle. If P_+ and P_- are the corresponding points on the $+,-$ edges of, say, the cycle B, then we may join them by the dotted path as in Fig. 22. Since this path can be deformed to A_1, we obtain the important identity

$$g(P_-) - g(P_+) = \int_{A_1}\omega \ . \tag{D1}$$

We conclude these preliminaries by observing that a holomorphic differential ω is automatically closed as a 1-form, i.e., $d\omega$ must be 0.

It is now easy to derive Riemann's bilinear relations. Let ω_J, ω_K be two elements of the homology basis, write $\omega_J = dg_J$ on the cut Riemann surface, and apply Green's theorem. The result is

$$0 = \int \omega_J \wedge \omega_K = \int d(g_J\omega_K) = \sum_{I=1}^h \int_{-B_I^+ \cup B_I^-} \int g_J\omega_K + \int_{-A_I^+ \cup A_I^-} g_J\omega_K$$
$$= \sum_{I=1}^h \oint_{A_I}\omega_J \oint_{B_I}\omega_K - \oint_{B_I}\omega_J \oint_{A_I}\omega_K = \Omega_{JK} - \Omega_{KJ} \ , \tag{D2}$$

showing that Ω is indeed symmetric. Next let $\omega = \sum_{J=1}^h c_J\omega_J$ by any holomorphic differential, which we again write as $\omega = dg$. The same arguments yield

$$0 < \frac{1}{2i}\int \bar{\omega} \wedge \omega = \frac{1}{2i}\int d(\bar{g}\omega) = \sum_{I=1}^h \int_{-B_I^+ \cup B_I^-} \bar{g}\omega + \int_{-A_I^+ \cup A_I^-} g\omega$$
$$= \sum_{I=1}^h \oint_{A_I}\bar{\omega} \oint_{B_I}\omega - \oint_{B_I}\bar{\omega} \oint_{A_I}\omega = \text{Im} \sum_{1 \le J,K \le h} \Omega_{JK} c_J \bar{c}_K \ , \tag{D3}$$

and the second Riemann bilinear relation is established.

We now provide a proof of Abel's theorem characterizing the divisors of meromorphic functions as the kernel of Abel's map. Let f be a meromorphic function on M, and let $z_1,..,z_k, w_1,\ldots,w_k$ be its zeros and poles. Then df/f is an Abelian differential of the third kind with simple poles and residues ± 1 at these points, and thus can be expressed as

$$df/f = \sum_{i=1}^n \omega_{z_i w_i} + \sum_{J=1}^h c_J\omega_J \ . \tag{D4}$$

Here ω_{z_i,w_i} are the normalized meromorphic differentials introduced in Sec. VI.F, and c_J are some complex scalar coefficients. Since the integral of df/f over any closed cycle must be a multiple of $2\pi i$, we deduce that

$$2\pi i n_K = \oint_{A_K} df/f = c_K \ ,$$
$$2\pi i m_K = \oint_{B_K} df/f = 2\pi i \sum_{j=1}^m \int_{w_j}^{z_j}\omega_K + \sum_{J=1}^h c_J\Omega_{JK} \ , \tag{D5}$$

for some integers n_K and m_K. This just means that $I(\sum_{j=1}^n z_j - \sum_{j=1}^n w_j)$ belongs to the lattice $\mathbf{Z}^h + \Omega\mathbf{Z}^h$. The converse has already been established via theta-function formulas, but we can also obtain it easily at this point by reversing the above arguments. Indeed if $I(\sum_{j=1}^n z_j - \sum_{j=1}^n w_j) \equiv 0$ then Eq. (D5) defines integers (n_K, m_K) out of which a differential ω can be constructed as in Eq. (D4) with periods multiples of $2\pi i$. In particular, $f(z) = \exp(\int_{z_0}^z \omega)$ is well defined on M and has the desired zeros and poles.

Finally we come to the zeros of the theta functions $\vartheta(\zeta,\Omega)$. Set

$$f(z)=\vartheta\left[\int_{z_0}^{z}\omega+\zeta,\Omega\right] . \tag{D6}$$

If f does not vanish identically as a function of z, df/f will be holomorphic away from the zeros of f, and hence we may apply Green's theorem to the cut Riemann surface with tiny disks S_i around the zeros of f removed:

$$\begin{aligned}0&=\int_{M_{\rm cut}\setminus US_i}d(df/f)\\&=-\sum\int_{\partial S_i}df/f+\sum_{J=1}^{h}\int_{-A_J^+ + A_J^-}df/f\\&\quad+\int_{-B_J^+\cup B_J^-}df/f .\end{aligned} \tag{D7}$$

The last term on the right-hand side is 0, since f is invariant under A periods. Under the B_K period df/f changes by $-2\pi i\omega_K$. Since the integrals over ∂S_i just produce $2\pi i$ (# of zeros of f), it follows that f has exactly h zeros.

To establish Eq. (6.37) we again use the cut Riemann surface with base point P (see Fig. 11) to write the Abelian differentials ω_K as $\omega_K=dg_K$. The jumps of g_K across A_L and B_L are then $-\Omega_{KL}$ and δ_{KL}, respectively. Green's theorem implies

$$\begin{aligned}0&=\int_{M\setminus US_i}d(g_K df/f)\\&=-\sum\int_{\partial S_i}g_K df/f+\sum_{L=1}^{h}\int_{(-B_L^+)\cup B_L^-}g_K df/f+\sum_{L=1}^{h}\int_{(-A_L^+)\cup A_L^-}g_K df/f\\&=-2\pi i\sum g(z_i)+\sum_{L=1}^{h}\delta_{KL}\int_{B_L}df/f-\sum_{L=1}^{h}\Omega_{KL}\int_{A_L}df/f+\sum 2\pi i\int_{A_L}g_K\omega_L+2\pi i\Omega_{KL}\int_{A_L}\omega_L\\&\equiv-2\pi i\sum g(z_i)+\left[-\pi i\Omega_{KK}-2\pi i\int_{z_0}^{P}\omega_K-2\pi i\zeta_K\right]+2\pi i\int_{A_L}g_K\omega_L\end{aligned} \tag{D8}$$

up to lattice points on $Z^h+\Omega Z^h$. Here P is the common point to all the basis cycles. Taking $P=z_0$, which is no loss of information, we recognize this relation as the desired relation, with the right definition of Δ as in Eq. (6.37).

It is now easy to deduce the zero set of $\vartheta(\zeta,\Omega)$ itself if we assume the characterization of those ζ for which f vanishes identically to be of the form $\zeta=I(w_1+\cdots+w_h)-\Delta$, where $w_1,\ldots,w_h$ is the set of poles of a nonconstant meromorphic function g. In fact, if ζ falls within this case, we may arrange for g to vanish at z_0. By Abel's theorem $I(\text{zeros of } g)\equiv I(w_1+\cdots+w_h)$. Substituting in the formula for ζ and noting that g must have zeros, one of which is z_0, gives $\zeta=I(z_1+\cdots+z_{h-1})-\Delta$. When z does not fall in this case, the first part of Riemann's vanishing theorem expresses ζ as $I(z_1+\cdots+z_h)-\Delta$, and again z_0 must be among these points.

APPENDIX E: THETA FUNCTIONS FOR THE TORUS

Ordinary theta functions, together with their properties, will be listed here. All four theta functions can be expressed in terms of a single one as translations thereof by a half-period:

$$\begin{aligned}&\vartheta_{00}(z,\tau)=\vartheta_3(z,\tau)=\vartheta(z,\tau) ,\\&\vartheta_{01}(z,\tau)=\vartheta_4(z,\tau)=\vartheta(z+\tfrac{1}{2},\tau) ,\\&\vartheta_{10}(z,\tau)=\vartheta_2(z,\tau)=e^{\pi i\tau/4+\pi iz}\vartheta(z+\tfrac{1}{2},\tau) ,\\&\vartheta_{11}(z,\tau)=\vartheta_1(z,\tau)=e^{\pi i\tau/4+\pi iz}\vartheta\left[z+\tfrac{1}{2}+\frac{\tau}{2},\tau\right] ,\end{aligned} \tag{E1}$$

where ϑ may be defined through

$$\vartheta(z,\tau)=\sum_{n=-\infty}^{\infty}e^{i\pi n^2+2\pi niz} . \tag{E2}$$

This series is absolutely convergent for $\text{Im}\tau>0$, and $\theta(z,\tau)$ is holomorphic in z. Under shifts by the periods, we have

$$\begin{aligned}&\vartheta_{ab}(z+1,\tau)=(-1)^a\vartheta_{ab}(z,\tau) ,\\&\vartheta_{ab}(z+\tau,\tau)=(-)^b e^{-\pi i\tau-2\pi iz}\vartheta_{ab}(z,\tau) ,\end{aligned} \tag{E3}$$

whereas shifts under half-periods produce

$$\begin{aligned}&\vartheta_{ab}(z+\tfrac{1}{2},\tau)=(-)^{ab}\vartheta_{a(b+a)}(z,\tau) ,\\&\vartheta_{ab}(z+\tfrac{1}{2}\tau,\tau)=(-i)^b e^{-\pi i\tau/4-\pi iz}\vartheta_{(a+b)b}(z,\tau) ,\end{aligned} \tag{E4}$$

where addition of a and b is understood modulo 2. Their next fundamental property is their behavior under modular transformations:

$$\begin{aligned}&\vartheta_{ab}(z,\tau+1)=e^{\pi ia/4}\vartheta_{a(b+a+1)}(z,\tau) ,\\&\vartheta_{ab}\left[\frac{z}{\tau},-\frac{1}{\tau}\right]=(-)^{ab}\sqrt{-i\tau}\,e^{\pi iz^2/\tau}\vartheta_{ba}(z,\tau) .\end{aligned} \tag{E5}$$

There also exist famous infinite product representations for these functions:

$$\vartheta_{0b}(z,\tau)=\prod_{n=1}^{\infty}(1-e^{2\pi in\tau})\left\{1-\exp\left[2\pi i\left[n\tau+z-\frac{1-b}{2}-\frac{\tau}{2}\right]\right]\right\}\left\{1-\exp\left[2\pi i\left[n\tau-z-\frac{1-b}{2}-\frac{\tau}{2}\right]\right]\right\},$$
$$\vartheta_{1b}(z,\tau)=i^{b}e^{\pi i\tau/4}[e^{i\pi z}+(-)^{b}e^{-i\pi z}]\times\prod_{n=1}^{\infty}(1-e^{2\pi in\tau})\left\{1-\exp\left[2\pi i\left[n\tau+z-\frac{1-b}{2}\right]\right]\right\}\left\{1-\exp\left[2\pi i\left[n\tau-z-\frac{1-b}{2}\right]\right]\right\}. \quad \text{(E6)}$$

Let us also mention an example of a Riemann identity:

$$\vartheta_{00}(x)\vartheta_{01}(y)\vartheta_{10}(u)\vartheta_{11}(v)-\vartheta_{00}(y)\vartheta_{01}(x)\vartheta_{10}(v)\vartheta_{11}(u)-\vartheta_{00}(u)\vartheta_{01}(v)\vartheta_{10}(x)\vartheta_{11}(y)+\vartheta_{00}(v)\vartheta_{01}(u)\vartheta_{10}(y)\vartheta_{11}(x)=2\vartheta_{00}(x_1)\vartheta_{01}(y_1)\vartheta_{10}(u_1)\vartheta_{11}(v_1), \quad \text{(E7)}$$

with

$$x_1=\tfrac{1}{2}(x+y+u+v),\quad y_1=\tfrac{1}{2}(x+y-u-v),$$
$$u_1=\tfrac{1}{2}(x-y+u-v),\quad v_1=\tfrac{1}{2}(x-y-u+v).$$

It may be reexpressed as

$$\sum_{a,b}(-1)^{a+b}\vartheta_{ab}(x)\vartheta_{ab}(y)\vartheta_{ab}(u)\vartheta_{ab}(v)=2\vartheta_{11}(x_1)\vartheta_{11}(y_1)\vartheta_{11}(u_1)\vartheta_{11}(v_1). \quad \text{(E7')}$$

Here we have suppressed the common τ dependence. In particular, upon setting $y=u=v=0$ one gets

$$\vartheta_{11}(x)\vartheta_{00}(o)\vartheta_{01}(o)\vartheta_{10}(o)=2\vartheta_{11}(\tfrac{1}{2}x)\vartheta_{00}(\tfrac{1}{2}x)\vartheta_{01}(\tfrac{1}{2}x)\vartheta_{10}(\tfrac{1}{2}x). \quad \text{(E8)}$$

For a list of Riemann identities, see Mumford (1983), Lecture I, pp. 16–23.

Also of importance are the "theta constants," obtained by setting $z=0$ in the above. As modular forms, they have profound significance in number theory. Especially well known is

$$\vartheta'_{11}(o,\tau)=-2\pi\eta(\tau)^3, \quad \text{(E9)}$$

where the Dedekind eta function is given by

$$\eta(\tau)=e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}). \quad \text{(E10)}$$

Under a modular transformation, $\eta(\tau)$ transforms according to the Jacobi rule with $\varepsilon^4=1$,

$$\eta\left[\frac{a\tau+b}{c\tau+d}\right]=\varepsilon(cz+d)^{1/2}\eta(\tau).$$

In addition one has the Jacobi relations

$$\vartheta'_{11}(o,\tau)=-\pi\vartheta_{00}(o,\tau)\vartheta_{01}(o,\tau)\vartheta_{10}(o,\tau),$$
$$\vartheta^4_{00}(o,\tau)=\vartheta^4_{01}(o,\tau)+\vartheta^4_{10}(o,\tau). \quad \text{(E11)}$$

We take the opportunity to recall the Poisson resummation formula,

$$\sum_{n=-\infty}^{\infty}e^{-\lambda 2\pi^2n^2}=\left[\frac{1}{2\pi\lambda}\right]^{1/2}\sum_{n=-\infty}^{\infty}e^{-n^2/2\lambda},$$

or more generally, if $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(n)=\int_{-\infty}^{\infty}dx\,e^{-inx}f(x),$$

then

$$\sum_{n\in Z}f(2\pi n)=\frac{1}{2\pi}\sum_{n\in Z}\hat{f}(n).$$

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