$$H_A{}^B=\mathcal{D}_A\delta V^B-\delta V^C T_{CA}{}^B+\delta V^C\Omega_C E_A{}^B \;. \tag{3.18}$$

(iii) Super Weyl transformations, forming a group sWeyl(M). These are generated by a real scalar superfield Σ,

$$\begin{aligned} E_M^a &= e^{\Sigma}\hat{E}\,{}_M^a \;,\\ E_M^\alpha &= e^{\Sigma/2}[\hat{E}\,{}_M^\alpha+\hat{E}\,{}_M^a(\gamma_a)^{\alpha\beta}\hat{\mathcal{D}}_\beta\Sigma]\;, \end{aligned} \tag{3.19}$$

and induce the following transformation laws on the superconnection, supercurvature, and superderivatives:

$$\begin{aligned} \Omega_M &= \hat{\Omega}_M+\hat{E}_M^a\varepsilon_a{}^b\hat{\mathcal{D}}_b\Sigma+\hat{E}\,{}_M^\alpha(\gamma_5)_\alpha{}^\beta\hat{\mathcal{D}}_\beta\Sigma\;,\\ R_{+-} &= e^{-\Sigma}(\hat{R}_{+-}-2i\hat{\mathcal{D}}_+\hat{\mathcal{D}}_-\Sigma)\;,\\ \mathcal{D}_+^n &= e^{(n-1/2)\Sigma}\hat{\mathcal{D}}\,{}_+^n e^{-n\Sigma}\;,\\ \mathcal{D}_-^n &= e^{-(n+1/2)\Sigma}\hat{\mathcal{D}}\,{}_-^n e^{+n\Sigma}\;. \end{aligned} \tag{3.20}$$

The infinitesimal form of Eq. (3.19) reads

$$\begin{aligned} &H_a{}^b=\delta\Sigma\delta_a{}^b,\quad H_\alpha{}^\beta=\tfrac{1}{2}\delta\Sigma\delta_\alpha{}^\beta\;,\\ &H_\alpha{}^a=0,\quad H_a{}^\alpha=(\gamma_a)^{\alpha\beta}\mathcal{D}_\beta\delta\Sigma\;. \end{aligned} \tag{3.21}$$

It will be useful to keep in mind that not all H's are independent due to the torsion constraints (3.11). The simplest set of independent deformations is $H_+{}^+$, $H_-{}^-$, and $H_\alpha{}^b$. The other components can then be calculated using the torsion constraints. To first order in H, we have the general formula

$$\begin{aligned} \delta T_{AB}{}^C = &-H_A{}^D T_{DB}{}^C+T_{AB}{}^D H_D{}^C+(-)^{ab}H_B{}^D T_{DA}{}^C\\ &-\mathcal{D}_A H_B{}^C+(-)^{ab}\mathcal{D}_B H_A{}^C+\psi_A E_B{}^C\\ &-(-)^{ab}\psi_B E_A{}^C\;, \end{aligned}$$

where $\psi_A=E_A{}^M\delta\Omega_M$. These imply

$$\begin{aligned} &H_z^z=\mathcal{D}_+H_+{}^z+2H_+{}^+\;,\\ &H_z{}^{\bar z}=\mathcal{D}_+H_+{}^{\bar z}\;,\\ &H_+{}^-=-\tfrac{1}{2}\mathcal{D}_+H_-{}^{\bar z}-\tfrac{1}{2}\mathcal{D}_-H_+{}^{\bar z}\;,\\ &H_z{}^-=\mathcal{D}_+H_+{}^-+\frac{i}{2}R_{+-}H_+{}^{\bar z}\;,\\ &H_z^+=\mathcal{D}_+H_+{}^++\mathcal{D}_+H_-{}^-+\mathcal{D}_-H_+{}^-+\frac{i}{2}R_{+-}H_-{}^{\bar z}\;,\\ &\delta\Omega_+=-i\mathcal{D}_{\bar z}H_+{}^{\bar z}-iR_{+-}H_-{}^{\bar z}+\tfrac{1}{2}(\mathcal{D}_-H_+{}^{\bar z})\Omega_--H_+{}^{\bar z}\Omega_{\bar z}\;. \end{aligned} \tag{3.22}$$

2. Supercomplex structures

By analogy with two-dimensional geometry we introduce a supercomplex structure

$$J_M{}^N=E_M{}^a\varepsilon_a{}^bE_b{}^N+E_M{}^\alpha(\gamma_5)_\alpha{}^\beta E_\beta{}^N\;, \tag{3.23}$$

which is a super-reparametrization tensor, and a local U(1) scalar. The main properties of $J_M{}^N$ are

$$J_M{}^NJ_N{}^P=-\delta_M{}^P \tag{3.24}$$

and the fact that it depends only on the *superconformal class* of $E_A{}^M$, i.e., it is invariant under the super Weyl transformations of (iii).

The almost complex structure $J_M{}^N$ of Eq. (3.23) may be used to define complex or, in this case, superholomorphic coordinates on the surface, provided this almost complex structure is integrable. This is actually a consequence of the superconformal flatness of two-dimensional supergeometry. A direct check of integrability illustrating the role of the torsion constraints is obtained by introducing the following one-forms:

$$\begin{aligned} \zeta^M &= dz^M-i\,dz^NJ_N{}^M\;,\\ \bar\zeta^M &= dz^M+i\,dz^NJ_N{}^M\;. \end{aligned} \tag{3.25}$$

ζ^M by itself has only two independent components, in view of Eq. (3.25). The almost complex structure $J_M{}^N$ is integrable provided

$$d\zeta^M\equiv 0 \pmod{\zeta^N}\;. \tag{3.26}$$

Using the explicit expression for $J_M{}^N$ in Eq. (3.23), as well as the definition of the torsion $T_{BC}{}^A$ of the $N=1$ supergeometry, we get

$$\begin{aligned} d\zeta^M= &-\tfrac{1}{2}\bar\zeta^PE_P{}^{\bar z}\zeta^QE_Q{}^-(T_{-\bar z}{}^+E_+{}^M+T_{-\bar z}{}^zE_z{}^M)\\ &-\tfrac{1}{4}\bar\zeta^PE_P{}^-\bar\zeta^QE_Q{}^-(T_{--}{}^+E_+{}^M+T_{--}{}^zE_z{}^M)\\ &\pmod{\zeta^N}\;, \end{aligned} \tag{3.27}$$

which indeed yields Eq. (3.26) with the help of the torsion constraints (3.11) and their consequences (3.12). Conversely, a supergeometry will support a complex structure only when the above torsion constraints are satisfied.

Thus we may define superholomorphic and superantiholomorphic functions by

$$J_M{}^N\mathcal{D}_Nf=i\mathcal{D}_Mf,\quad J_M{}^N\mathcal{D}_N\bar f=-i\mathcal{D}_M\bar f\;, \tag{3.28}$$

or, equivalently,

$$\mathcal{D}_-f=0,\quad \mathcal{D}_+\bar f=0\;.$$

The supersurface together with a supercomplex structure $J_M{}^N$ will be called a super Riemann surface, although strictly speaking the geometry of the supersurface is not Riemannian, i.e., there is no metric for superspace. One can verify that a super Riemann surface admits an atlas of coordinate patches whose transition functions are superholomorphic. This approach provides an alternative definition of a super Riemann surface.

3. Flat and conformally flat superspace

Flat $N=1$ superspace is given by the superzweibein

$$\begin{aligned} &E_m{}^a=\delta_m{}^a,\quad E_m{}^\alpha=0\;,\\ &E_\mu{}^a=(\gamma^a)_\mu{}^\beta\theta_\beta,\quad E_\mu{}^\alpha=\delta_\mu{}^\alpha\;, \end{aligned} \tag{3.29}$$

and the superderivatives take the simple form

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}, \quad \mathcal{D}_- = \frac{\partial}{\partial\bar\theta} + \bar\theta\frac{\partial}{\partial\bar z} . \tag{3.30}$$

Equivalently, flat superspace is characterized by $R_{+-}=0$. Locally every supergeometry is superconformal [i.e., equivalent under a super Weyl and local U(1) transformation] to flat supergeometry. One can easily see this directly from the equations characterizing super-reparametrizations and by using the analogous result for ordinary geometry, or by evaluating the supercomplex structure tensor $J_M{}^N$ of Eq. (3.23). Locally, then, Eq. (3.28) is solved by

$$f = f_0(z) + \theta f_1(z) ,$$

where f_0 and f_1 are holomorphic in the ordinary sense.

Globally, however, there may be topological obstructions, and it will be necessary to introduce supermoduli space, i.e., the space of inequivalent superconformal structures. We shall take up this issue in Secs. III.E and III.G.

A complete local analysis of $N=1$ two-dimensional supergravity is due to Howe (1979). In particular, the fact that any two-dimensional supergeometry is locally superconformally flat is due to him. The superfield formalism and variations $H_A{}^B$ were used by Martinec (1983) to compute the super Weyl anomaly. The supercomplex structure $J_M{}^N$ was introduced by D'Hoker and Phong (1987a), who also showed that its integrability (vanishing of the Nijenhuis tensor) is a consequence of the Wess-Zumino torsion constraints. Interpretations of Wess-Zumino constraints as reductions of G structures were subsequently given by Giddings and Nelson (1988a).

The alternative approach to super Riemann surfaces through superholomorphic function theory and charts as in Eq. (3.28) was developed by Friedan (1986), Baranov, Frolov, and Schwarz (1987), and Crane and Rabin (1987). That the two classes of super Riemann surfaces coincide was proved by Giddings and Nelson (1988a).

C. Component formalism for $N=1$ supergravity

To obtain a better understanding of supergeometry and supergravity, it should be useful to discuss the associated component formulation. The passage from the superfield to the component language requires the elimination of the auxiliary fields required by the superfields. This is usually accomplished by fixing the Wess-Zumino gauge for the superzweibein. This gauge is defined by the condition that in the expansion in powers of θ we have[11]

$$\begin{aligned} &E_\mu{}^\alpha \sim \delta_\mu{}^\alpha + \theta^\nu e^{*\alpha}_{\nu\mu}, \quad E_\mu{}^a \sim \theta^\nu e^{**a}_{\nu\mu} , \\ &e^{*\alpha}_{\nu\mu} = e^{*\alpha}_{\mu\nu}, \quad e^{**a}_{\nu\mu} = e^{**a}_{\mu\nu} , \end{aligned} \tag{3.31}$$

up to higher-order terms. A superzweibein can always be brought to this gauge by a super-reparametrization, which is obtained through algebraic equations alone. The main ingredients of the supergeometry can then be derived from the Bianchi identities and the torsion constraints. The results for their full θ expansions are

$$\begin{aligned} &E_m{}^a = e_m{}^a + \theta\gamma^a\chi_m - \frac{i}{2}\theta\bar\theta e_m{}^a A , \\ &E_m{}^\alpha = -\tfrac{1}{2}\chi_m{}^\alpha - \frac{i}{2}\theta^\beta(\gamma_m)_\beta{}^\alpha A - \tfrac{1}{2}\theta^\beta(\gamma_5)_\beta^\alpha\omega_m \\ &\qquad + i\theta\bar\theta[\tfrac{1}{2}(\gamma_m)^{\alpha\beta}\Lambda_\beta - \tfrac{3}{8}\chi_m^\alpha A] , \\ &E_\mu{}^a = (\gamma^a)_\mu{}^\beta\theta_\beta , \\ &E_\mu{}^\alpha = \delta_\mu{}^\alpha(1 + i\theta\bar\theta A/4) , \\ &\mathrm{sdet} E_M{}^A = e\left[1 + \tfrac{1}{2}\theta\gamma^n\chi_n - \frac{i}{2}\theta\bar\theta A + \tfrac{1}{8}\theta\bar\theta\varepsilon^{mn}\chi_m\gamma_5\chi_n\right] , \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} &\omega_m = -e_m{}^a\varepsilon^{pq}\partial_p e_q{}^a - \tfrac{1}{2}\chi_m\gamma_5\gamma^p\chi_p , \\ &\Lambda = -i\gamma_5\varepsilon^{mn}D_m\chi_n - \tfrac{1}{2}\gamma^m\chi_m A , \\ &D_m\chi_n = \partial_m\chi_n + \tfrac{1}{2}\omega_m\gamma_5\chi_n . \end{aligned} \tag{3.33}$$

Notice that since $E_\mu{}^\alpha$ is basically the Kronecker symbol between a μ and an α index, the distinction between U(1) and Einstein spinor indices is lost in Wess-Zumino gauge, and θ may be written either with α or μ indices and transforms as a spinor under U(1). It is also useful to record the spinor components of the inverse superzweibeins,

$$\begin{aligned} &E_\alpha{}^\mu = \delta_\alpha{}^\mu + \tfrac{1}{2}\theta^\gamma\gamma^m_{\alpha\gamma}\chi_m{}^\mu + i\theta\bar\theta\hat{e}_\alpha{}^\mu , \\ &E_\alpha{}^m = \theta^\beta\gamma^m_{\alpha\beta} + \tfrac{1}{2}\theta\bar\theta(\gamma^n\gamma^m)_\alpha{}^\gamma\chi_{n\gamma} , \end{aligned} \tag{3.34}$$

where

$$\hat{e}^{\,\mu}_{\alpha} = \tfrac{1}{4}\delta_\alpha{}^\mu A + \frac{i}{2}(\gamma^m\gamma_5)_\alpha{}^\mu\omega_m - \frac{i}{4}(\gamma^n\gamma^m)_\alpha{}^\gamma\chi_{n\gamma}\chi_m{}^\mu . \tag{3.35}$$

For the superconnection and supercurvature we have

$$\begin{aligned} &\Omega_m = \omega_m + \frac{i}{2}\theta\gamma_5\chi_m A - i\theta\gamma_5\gamma_m\Lambda + i\theta\bar\theta\hat\omega_m , \\ &\Omega_\mu = \frac{i}{2}(\gamma_5)_\mu{}^\beta\theta_\beta A , \\ &R_{+-} = A + \theta^\alpha\Lambda_\alpha + i\theta\bar\theta C , \\ &C = R + \frac{i}{2}\chi_a\gamma^a\Lambda + \frac{i}{8}\varepsilon^{ab}\chi_a\gamma_5\chi_b A + \tfrac{1}{2}A^2 , \end{aligned} \tag{3.36}$$

where

$$\hat\omega_n = -\tfrac{1}{2}A\omega_n - \tfrac{1}{2}\chi_m\gamma_5\gamma_n\gamma^m\Lambda - \tfrac{1}{2}e_n^a\varepsilon_a{}^b e_b{}^m\partial_m A ,$$

with R the curvature of the connection ω_m appearing in Eq. (3.33):

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[11]This is the correct choice provided—as we have—the gamma matrices are taken to be symmetric.

$$R = \varepsilon^{mn}\partial_m \omega_n \ . \tag{3.37}$$

Thus the supergravity multiplet $E_M{}^A$ reduces to a zweibein $e_m{}^a$, a gravitino field $\chi_m{}^\alpha$, and an auxiliary field A which will not appear in the component Lagrangian. Wess-Zumino gauge is left invariant under a subgroup of all super-reparametrizations and local U(1) transformations, given by

$$\delta V^m = \delta v^m - \theta\gamma^m\zeta - \tfrac{1}{2}\theta\bar\theta\chi_n\gamma^m\gamma^n\zeta \ ,$$

$$\delta V^\mu = \zeta^\mu + \tfrac{1}{2}\theta^\alpha(\gamma_5)_\alpha{}^\mu l - \tfrac{1}{2}\theta\gamma^n\zeta\chi_n{}^\mu + i\theta\bar\theta\hat\zeta^\mu \ ,$$

$$L = l - \frac{i}{2}A\theta\gamma_5\zeta + \omega_n\theta\gamma^n\zeta + i\theta\bar\theta\hat l \ ,$$

where we have used the abbreviations

$$\hat\zeta = \frac{i}{2}\gamma_5\gamma^n\zeta\omega_n + \frac{i}{4}\chi_n(\chi_m\gamma^n\gamma^m\zeta) - \tfrac{1}{2}\zeta A \ ,$$

$$\hat l = \frac{1}{2}\zeta\gamma_5\Lambda - \frac{i}{2}\omega_n\chi_m\gamma^n\gamma^m\zeta + \tfrac{1}{4}\zeta\gamma_5\gamma^n\chi_n A \ .$$

It is now straightforward to translate the symmetries of the superzweibein into component language as well. The super-reparametrizations relevant to the component language are those that preserve the Wess-Zumino gauge up to local U(1) and super Weyl transformations. They decompose into reparametrization invariance and an $N=1$ supersymmetry. Super Weyl transformations will take us out of this gauge, so the component transformations written below are obtained only after compensation by a super-reparametrization and a local U(1) transformation taking us back to Wess-Zumino gauge.

(i) Local U(1) symmetry forming the group sU(1):

$$\delta e_m{}^a = l\varepsilon^a{}_b e_m{}^b \ ,$$

$$\delta\chi_m = -\tfrac{1}{2}l\gamma_5\chi_m \ ,$$

$$\delta A = 0 \ ,$$

$$\delta\omega_m = \partial_m l \ .$$

(ii) Reparametrizations, forming Diff(M):

$$\delta e_m{}^a = \delta v^n\partial_n e_m{}^a + e_n{}^a\partial_m\delta v^n \ ,$$

$$\delta\chi_m = \delta v^n\partial_n\chi_m + \chi_n\partial_m\delta v^n \ ,$$

$$\delta A = \delta v^n\partial_n A \ ,$$

$$\delta\omega_m = \delta v^n\partial_n\omega_m + \omega_n\partial_m\delta v^n \ .$$

(iii) Local $N=1$ supersymmetry:

$$\delta e_m{}^a = \zeta\gamma^a\chi_m \ ,$$

$$\delta\chi_m = -2D_m\zeta - iA\gamma_m\zeta \ ,$$

$$\delta A = \zeta\Lambda \ ,$$

$$\delta\omega_m = i\zeta\gamma_m\gamma_5\Lambda + \frac{i}{2}\zeta\gamma_5\chi_m A \ ,$$

$$\delta\Lambda = -\tfrac{1}{2}\gamma^m\zeta(\partial_m A + \tfrac{1}{2}\chi_m\Lambda) - i\zeta C \ ,$$

$$\delta C = \frac{i}{2}(\chi_p\gamma^m\gamma^p\zeta)(\partial_m A + \tfrac{1}{2}\chi_m\Lambda) - i\zeta\gamma^m D_m\Lambda - \tfrac{1}{2}\zeta\gamma^p\chi_p C - \tfrac{1}{2}A\zeta\Lambda \ .$$

(iv) Weyl transformations, forming the group Weyl(M):

$$\delta e_m{}^a = \delta\sigma e_m{}^a \ ,$$

$$\delta\chi_m = \tfrac{1}{2}\delta\sigma\chi_m \ .$$

(v) Super Weyl scalings:

$$\delta e_m{}^a = 0 \ ,$$

$$\delta\chi_m = \gamma_m\delta\lambda \ .$$

Finally we note that the "super Euler number" reduces to the standard Euler number

$$\chi(M) = \frac{i}{2\pi}\int_M d^2\mathbf{z}\, ER_{+-} = \frac{1}{2\pi}\int_M d^2\xi\sqrt{g}R \ , \tag{3.38}$$

where the volume element on the superworldsheet is given by

$$d^2\mathbf{z}\,E = d^2\xi\, d\theta\, d\bar\theta\, \mathrm{sdet}E_M{}^A \ . \tag{3.39}$$

The topology of the super Riemann surface is just that of its "body" component when it is viewed as a De Witt (1983) supermanifold, and hence the topological classification is again by the number of handles, when the surface has no boundaries.

The passage to Wess-Zumino gauge and the construction of the super Weyl symmetry has been carried out by Howe (1979). The formulas of Howe have been reproduced here in Euclidean signature for convenience.

D. Path integrals for the RNS superstring

The superspace action for the Ramond-Neveu-Schwarz string model is obtained by coupling scalar "position" superfields X^μ, $\mu=1,\ldots,d=10$ to two-dimensional $N=1$ supergravity. The matter action is then given by

$$I_m = \frac{1}{8\pi}\int d^2\mathbf{z}\,E\mathcal{D}^\alpha X^\mu\mathcal{D}_\alpha X_\mu + \lambda\chi(M)$$
$$= \frac{1}{4\pi}\int d^2\mathbf{z}\,E\mathcal{D}_- X^\mu\mathcal{D}_+ X_\mu + \lambda\chi(M) \ . \tag{3.40}$$

We may decompose X^μ into components: $X^\mu = x^\mu + \theta^\alpha\psi^\mu_\alpha + i\theta\bar\theta F^\mu$, where x^μ and ψ^μ may be identified with the fields occurring in Eq. (3.1) and F^μ is an auxiliary field. The action (3.40) actually coincides with Eq. (3.1) in Wess-Zumino gauge except for a term F^2. The symmetries (i), (ii), (iii) of Sec. III.B of supergravity will become symmetries of I_m when X^μ is assigned the corresponding transformation laws: X^μ is a local U(1), super Weyl, and super-reparametrization scalar. In addition, I_m is evidently invariant under space-time Poincaré transformations if the target space-time is flat Minkowskian. Imposing the above symmetries, we find that the action (3.40) is unique.

To quantize the theory we integrate e^{-I_m} over all supergeometries $(E_M{}^A, \Omega_M)$ satisfying the torsion constraints and over all superfields X^μ, and we sum over all possible topologies of the super Riemann surface. Recalling that this reduces to the sum over the number of handles just as in the bosonic case, we may conjecture the contribution to the partition function at h string loops,

$$Z_h = \int DE_M{}^A D\Omega_M DX^\mu \delta(T) \exp(-I_m[X^\mu, E_M{}^A]) , \tag{3.41}$$

with the topology of the worldsheet fixed at h handles. The delta function enforcing the torsion constraints [Eq. (3.11)] is denoted by $\delta(T)$. It involves only algebraic equations, which are linear in Ω_M, so that the Ω_M integral may be ignored once the torsion constraints have been enforced.

Similarly, scattering amplitudes are obtained by integrating the product of e^{-I_m} by a number of vertex operators, exactly as in the bosonic case. We shall not reproduce the corresponding formulas here.

The integral assumes the existence of a local U(1), super-reparametrization-invariant measure, not depending on derivatives. The unique choice for DX^μ comes from the metric

$$\|\delta X^\mu\|^2 = \int_M d^2\mathbf{z}\, E\, \delta X^\mu \delta X_\mu . \tag{3.42}$$

We can carry out the integration over X^μ in Eq. (3.41) since it is Gaussian. As will be shown in Sec. III.E, however, the operator $\mathcal{D}_+\mathcal{D}_-^{(0)} = \Box_0$ has zero modes. First, there is the constant superfield corresponding to constant x^μ. For odd-spin structure, there will also be a Dirac zero mode for ψ^μ, but how many zero modes remain for $\Box_0$ may depend on the superconformal class. If there are odd zero modes of $\Box_0$ (analogous to Dirac zero modes), then the partition function of Eq. (3.41) must vanish—though of course correlation functions may be nonvanishing. Thus the proper formula is obtained by omitting only the constant zero mode, so that we obtain

$$Z_h = \Omega \int DE_M{}^A D\Omega_M \delta(T) \left[\frac{8\pi^2}{\int d^2\mathbf{z}\, E} \operatorname{sdet}'\Box_0 \right]^{-d/2} . \tag{3.43}$$

Here Ω is the volume of space-time, and the prime denotes omission of the translation zero mode. Note that the superfield X^μ depends on the spin structure, and hence so does the superdeterminant.

The integration over supergeometries is considerably more complicated. Since there are torsion constraints, we have the choice of using the first- or second-order formalisms (see de Witt and Freedman, 1983). In the first-order formalism, all 16 components of $E_M{}^A$ and all 4 of Ω_M are integrated over, subject to the torsion constraints, which may be represented by the use of Lagrange multipliers. Alternatively, in the second-order formalism, dependent degrees of freedom are completely eliminated by use of the torsion constraints, and the integration measure is restricted to the independent components only. Though elimination of dependent degrees of freedom can conveniently be achieved only if simultaneously a gauge condition is imposed (like Wess-Zumino gauge), the dependent infinitesimal variations of the supergeometry are easily determined, as was done in Eq. (3.22). To write down the metric on the space of supergeometries, infinitesimal variations are all that is needed; thus to construct the natural measure on the independent components of $\delta E_M{}^A$ and $\delta\Omega_M$, we recall that the only independent components of $H_A{}^B = E_A{}^M \delta E_M{}^B$ are $H_\alpha{}^a$, $\gamma_5 H = (\gamma_5)_\beta{}^\alpha H_\alpha{}^\beta$, and $H_\alpha{}^b$, the other components being given by Eq. (3.22). The expression for $\delta\Omega_M$ can be determined from Eq. (3.14). Note that these relations involve superderivatives of the independent components. Thus, in order to obtain a metric consistent with locality on the worldsheet, it is necessary to construct it in terms of independent fields only. This metric on $H_A{}^B$ should be of the form

$$\|\delta E_M{}^A\|^2 = \int d^2\mathbf{z}\, E[\varepsilon^{\alpha\beta} H_\alpha{}^a H_\beta{}^a + c_1 H_\alpha{}^a H_\beta{}^\beta + c_2(\gamma_5 H)(\gamma_5 H)] , \tag{3.44}$$

where c_1 and c_2 are undetermined numerical constants, analogous to c in Eq. (2.21). The measure on $DE_M{}^A$ will always be understood as coming from this metric. Associated with Eq. (3.44) is a quadratic form, constructed in the standard way, and denoted by $\langle H_1 \mid H_2 \rangle$.

Though super-reparametrization and local U(1) invariant, $\|\delta E_M{}^A\|$ fails to be super Weyl invariant, which will give rise to the super Weyl anomaly, as we shall see later on. Super Weyl invariance is recovered for the full amplitude, as the anomaly from the matter determinants and Faddeev-Popov ghosts cancel in the critical dimension $d=10$ and in the case of the heterotic string for gauge groups of rank 16. The same will hold true for possible (perturbative) gravitational and holomorphic anomalies arising in connection with the chiral Dirac determinants, as will be shown in Sec. VII. Of course, as higher string loop effects are considered and surfaces of nontrivial topology are used, there may be global reparametrization (or modular) anomalies. In the case of heterotic strings, for example, they give rise to further restriction to the gauge group $\mathrm{Spin}(32)/Z_2$ and $\mathrm{E}_8 \times \mathrm{E}_8$.

After all these symmetry groups have been factored out, we should be left with a (finite-dimensional) integral over the space of supergeometries that are inequivalent under any of these transformations, and we are now going to identify this space, first locally in Sec. III.E and then globally in Sec. III.G.

E. Deformations of supercomplex structures

The effect on $H_A{}^B$ of combined super-reparametrization δV^M and U(1) and super Weyl transformations δL and $\delta\Sigma$ is completely described by the action on the independent components of $H_A{}^B$ which were identified in Sec. III.B:

$$H_\alpha{}^\alpha = \delta\Sigma + \mathcal{D}_\alpha \delta V^\alpha ,$$
$$(\gamma_5)_\alpha{}^\beta H_\beta{}^\alpha = \delta L + (\gamma_5)_\alpha{}^\beta \mathcal{D}_\beta \delta V^\alpha - \delta V^C \Omega_C , \qquad (3.45)$$
$$H_\alpha{}^b = \mathcal{D}_\alpha \delta V^b + 2(\gamma^b)_{\alpha\gamma}\delta V^\gamma .$$

This shows that $H_\alpha{}^\alpha$ and $\gamma_5 H$ can be completely eliminated without any topological obstruction through a super Weyl and local U(1) transformation. Since anything proportional to a γ matrix can also be eliminated from $H_\alpha{}^b$ in a purely algebraic fashion, it is natural to introduce

$$(\mathcal{P}_1 \delta V)_\alpha{}^b = -(\gamma_c \gamma^b)_\alpha{}^\beta \mathcal{D}_\beta \delta V^c \qquad (3.46)$$

in analogy with Eq. (2.23). Upon isolating the various components we obtain

$$(\mathcal{P}_1 \delta V)_-{}^z = \mathcal{D}_- \delta V^z , \quad (\mathcal{P}_1 \delta V)_-{}^{\bar z} = 0 , \qquad (3.47)$$

and their complex-conjugate expressions. We observe that the only nonremovable H's are those $H_\alpha{}^b$'s not in the range of $\mathcal{P}_1$. At this stage in the bosonic case, we concluded that the metric deformations δg_{mn} not in the range of P_1 must belong to the orthogonal complement of the range of P_1. This step assumes that the metric $\|\delta g\|^2$ is nondegenerate and (positive) definite.

For the superstring case, we see that the metric defined in Eq. (3.44) is nondegenerate but fails to be definite (i.e., there exist $H \neq 0$ with $\|H\| = 0$). When the metric is nondefinite, there may in general be elements belonging to both Range$\mathcal{P}_1$ and $(\text{Range}\mathcal{P}_1)^\perp$, and the sum of these two spaces need not span the full space of deformations $H_\alpha{}^b$.

To analyze the structure of the complement of Range$\mathcal{P}_1$, let us investigate the intersection of Range$\mathcal{P}_1$ and $(\text{Range}\mathcal{P}_1)^\perp$. Introducing the natural metric

$$\|\delta V\|^2 = \int_M d^2 z\, E\, \delta V^\alpha \delta V_\alpha \qquad (3.48)$$

on the space of tensor fields of weight $n \oplus -n$, we readily derive the identity

$$(\text{Range}\mathcal{P}_1)^\perp = \text{Ker}\mathcal{P}_1^\dagger , \qquad (3.49)$$

where

$$(\mathcal{P}_1^\dagger H)^a = (\gamma_b \gamma^a)^{\beta\alpha} \mathcal{D}_\beta H_\alpha{}^b . \qquad (3.50)$$

Now assume that $H \in (\text{Range}\mathcal{P}_1) \cap (\text{Range}\mathcal{P}_1)^\perp$, then we have with the help of Eq. (3.49) that $H = \mathcal{P}_1 \delta V$ and $\mathcal{P}_1^\dagger H = 0$. Combining both, we obtain $\mathcal{P}_1^\dagger \mathcal{P}_1 \delta V = 0$, and so there must be an element δV *not* in Ker$\mathcal{P}_1$ which belongs, however, to Ker$\mathcal{P}_1^\dagger \mathcal{P}_1$. Conversely, if the kernels are equal, then such elements $\delta V \neq 0$ can belong to Ker$\mathcal{P}_1^\dagger \mathcal{P}_1$, and the intersection between Range$\mathcal{P}_1$ and (Range $\mathcal{P}_1)^\perp$ must be trivial:

$$\text{Ker}\mathcal{P}_1 = \text{Ker}\mathcal{P}_1^\dagger \mathcal{P}_1 \Longleftrightarrow (\text{Range}\mathcal{P}_1) \cap (\text{Range}\mathcal{P}_1)^\perp = \{0\} . \qquad (3.51)$$

Equivalently, this means that the inner product $\langle \ | \ \rangle$ remains nondegenerate upon restriction to Range$\mathcal{P}_1$. Consider the element $H_1 = \mathcal{P}_1 \delta V_1$ and $H_2 = \mathcal{P}_1 \delta V_2$ of Range$\mathcal{P}_1$, and compute their inner product:

$$\langle H_1 | H_2 \rangle = \langle \mathcal{P}_1 \delta V_1 | \mathcal{P}_1 \delta V_2 \rangle = \langle \delta V_1 | \mathcal{P}_1^\dagger \mathcal{P}_1 \delta V_2 \rangle . \qquad (3.52)$$

If this inner product vanishes for all δV_1, then $\delta V_2 \in \text{Ker}\mathcal{P}_1^\dagger \mathcal{P}_1$, by nondegeneracy of $\langle \ | \ \rangle$ on the space of all δV's. If the inner product $\langle \ | \ \rangle$ is to remain nondegenerate upon restriction to Range$\mathcal{P}_1$, then we must also have $H_2 = 0$, so that (3.51) holds. Thus the issue here is the relation between Ker$\mathcal{P}_1$ and Ker$\mathcal{P}_1^\dagger \mathcal{P}_1$. As will become clear during our subsequent discussion, the case of the torus is truly exceptional, and we shall treat it separately later on.

For $h \geq 2$ and $h = 0$, it will be shown in Sec. III.F that Ker$\mathcal{P}_1$ = Ker$\mathcal{P}_1^\dagger \mathcal{P}_1$, so that the intersection between Range$\mathcal{P}_1$ and $(\text{Range}\mathcal{P}_1)^\perp$ is the null vector only and the sum of Range$\mathcal{P}_1$ and $(\text{Range}\mathcal{P}_1)^\perp$ spans the full space of γ-traceless $H_\alpha{}^b$'s. Putting everything together we obtain the orthogonal decomposition

$$\{H_A{}^B\} = \{\delta\Sigma\} \oplus \{\delta L\} \oplus \text{Range}\mathcal{P}_1 \oplus \text{Ker}\mathcal{P}_1^\dagger . \qquad (3.53)$$

The elements of Ker$\mathcal{P}_1$ will be termed *superconformal Killing vectors* and those of Ker$\mathcal{P}_1^\dagger$ *super moduli deformations* or *holomorphic superquadratic differentials.*

To gain further insight into the nature of the super moduli deformations of Ker$\mathcal{P}_1^\dagger$, we rewrite $\mathcal{P}_1^\dagger$ componentwise

$$(\mathcal{P}_1^\dagger \Phi)^z = \mathcal{D}_+ \Phi^z_- , \quad (\mathcal{P}_1^\dagger \Phi)^{\bar z} = \mathcal{D}_- \Phi^{\bar z}_+ , \qquad (3.54)$$

and make contact with Wess-Zumino gauge by setting

$$\Phi_+{}^{\bar z} = \phi_0 + \theta\phi_+ + \bar\theta\phi_- + i\theta\bar\theta\phi_1 . \qquad (3.55)$$

The result is

$$(\mathcal{P}_1^\dagger \Phi)^{\bar z} = \phi_- + \theta\left[-i\phi_1 + \frac{3i}{4}A\phi_0\right] + \bar\theta(D_{\bar z}\phi_0 + \tfrac{1}{2}\chi_{\bar z}{}^+\phi_+) + \theta\bar\theta\left[-D_{\bar z}\phi_+ - \tfrac{1}{2}\chi_{\bar z}{}^+ D_z\phi_0 + \frac{3i}{2}\Lambda_-\phi_0 + iA\phi_- + \tfrac{1}{4}\chi_z{}^-\chi_{\bar z}{}^+\phi_-\right] . \qquad (3.56)$$

The changes $\delta E_M{}^A$ solving these equations will in general take us out of Wess-Zumino gauge, and a compensating super-reparametrization and U(1) transformation is needed, which, however, will not change the number of supermoduli. Under the hypothesis that the space of inequivalent supergeometries (to be termed supermoduli space later) is a supermanifold, we can determine its dimension at any point, and in particular at $(e_m{}^a, \chi_m)$ satisfying $\chi_z{}^- = 0$ and $A = 0$, so that

$$P_1^\dagger \phi_+ = 0 \quad \text{and} \quad P_{1/2}^\dagger \phi_0 = 0 \qquad (3.57)$$

where P_1, $P_{1/2}$ are the operators (2.48) familiar from the component formalism. The index theorem and a simple counting of the number of conformal Killing vectors and spinors in each case yield the dimension of the vector spaces $\mathrm{Ker}P_1^\dagger$ and $\mathrm{Ker}P_{1/2}^\dagger$ [cf. Eqs. (2.50) and (2.51)],

$$\dim \mathrm{Ker}\mathcal{P}_1^\dagger = \begin{cases}(0,0), & h=0,\\ (6h-6,4h-4), & h\geq 2.\end{cases} \tag{3.58}$$

Here the two integers denote, respectively, the dimensions for the even and the odd coordinates. More generally, operators $\mathcal{P}_n$ acting on superfields of arbitrary weight $n>0$ can be introduced and expressed in terms of the U(1)-covariant derivatives $\mathcal{D}_-^n$. In Wess-Zumino gauge they will admit expansions similar to Eq. (3.56) [see Eq. (3.66) below], and the previous arguments will show that the number of zero modes is given by

$$\dim(\mathrm{Ker}\mathcal{P}_n) = \begin{cases}(4n+2,4n), & h=0,\\ (0,0), & h\geq 2,\end{cases}$$

$$\dim(\mathrm{Ker}\mathcal{P}_n^\dagger) = \begin{cases}(0,0), & h=0,\\ ((4n+2)(h-1),4n(h-1)), & h\geq 2.\end{cases}$$

For the case of the torus with $h=1$, it will be clear that the arguments given in Sec. III.F in support of the direct sum decomposition of Eq. (3.53) break down. In short, the reason is that the natural choice for constant curvature on the torus is zero curvature, so that the auxiliary field A vanishes and (3.51) does not hold. Actually, the natural metric $\|H\|$ becomes degenerate on the torus. Thus we would like to analyze the supermoduli problem in a way that does not depend on this metric. Ultimately we are interested in describing and parametrizing those geometries which cannot be interrelated by super-reparametrizations, local U(1), or super Weyl transformations, and we shall now attack this issue directly.

We start by considering the full supergeometry with the torsion constraints. First, by a super Weyl transformation, we fix the curvature R_{+-} to zero; the fact that this can always be done will be shown in Sec. III.F. For the torus, $R_{+-}=0$ cannot be chosen in a unique way since this slice is left invariant under harmonic super Weyl scalings satisfying

$$\mathcal{D}_+\mathcal{D}_-^{(0)}\Sigma_0=0 \ . \tag{3.59}$$

The condition $R_{+-}=0$ is super-reparametrization and local U(1) invariant, and this is exactly what is needed to fix Wess-Zumino gauge, which we now do. In components, the zero-curvature condition becomes

$$\begin{aligned} &A=0\ ,\\ &\Lambda=-i\gamma_5\varepsilon^{mn}D_m\chi_n=0\ ,\\ &C=R=0\ ,\end{aligned} \tag{3.60}$$

where the components of the curvature were introduced in Eq. (3.36). Note that in view of Eq. (3.37) the last condition implies

$$\partial_m\omega_n-\partial_n\omega_m=0\ , \tag{3.61}$$

where ω_m is the only nonvanishing component of the connection Ω_M. The remaining symmetries of this slice are now local supersymmetry, local U(1) invariance, and ordinary reparametrizations, whose actions were listed in Sec. III.C.

However, on this slice, the form of the infinitesimal versions of these transformations may be considerably simplified. One finds that the effect of a reparametrization v^n, a supersymmetry ζ, and a local U(1) transformation l is given by

$$\begin{aligned} &\delta e_m{}^a=D_m(v^ne_n{}^a)+\tilde{l}\varepsilon^a{}_be_m{}^b+\tilde{\zeta}\gamma^a\chi_m\ ,\\ &\delta\chi_m=-2D_m\tilde{\zeta}-\tfrac{1}{2}\tilde{l}\gamma_5\chi_m\ ,\\ &\delta\omega_m=\partial_m\tilde{l}\ ,\end{aligned} \tag{3.62}$$

where we have introduced special combinations of local U(1) and supersymmetry transformations, defined by

$$\begin{aligned} &\tilde{l}=l+v^n\omega_n\ ,\\ &\tilde{\zeta}=\zeta-\tfrac{1}{2}v^n\chi_n\ .\end{aligned}$$

The action of the combined three symmetries is particularly simple; in fact it is global and triangular in the following sense. The (modified) local U(1) transformation $\tilde{l}$ acts globally on all three fields in a well-known way. The supersymmetry $\tilde{\zeta}$ no longer acts on ω_m, in contrast with ζ itself. This implies that the supersymmetry also integrates to a global action on χ_m, since the connection $D_m=\partial_m+\frac{1}{2}i\omega_m$ is invariant under $\tilde{\zeta}$ transformations. Finally, ordinary reparametrizations act only on $e_m{}^a$, and again their global action may be exploited to choose a *global* gauge for the "supertorus." Since ω_m satisfies Eq. (3.61), local U(1) transformations $\tilde{l}$ will eliminate all degrees of freedom of ω_m, except for the constant ones. Note that constant $\tilde{l}$'s have not been used to do so. Thus ω_m is constant, and this is unchanged by supersymmetry transformations $\tilde{\zeta}$.

We model the torus by a square with sides of unit length and opposite sides identified. If we assume that not all components of ω_m are multiples of 2π, so that D_m acting on spinors has no zero modes, then all components of χ_m may be eliminated via supersymmetry transformations $\tilde{\zeta}$. Similarly, all components of $\delta e_m{}^a$ are eliminated and $e_m{}^a$ may be chosen constant. Then, however, we must have $\omega_m=0$ by its very definition in Eq. (3.33), which is in contradiction with the original assumption, and hence all components of ω_m must be multiples of 2π. By redefining all fields by multiplications by a simple function, we may set $\omega_m=0$ without modifying the original boundary conditions. At $\omega_m=0$, the remaining components[12] of $e_m{}^a$ and χ_m are easily found.

For even-spin structure, D_m has no zero modes on spin fields, and we may set $\chi_m=0$ by supersymmetry and $e_m{}^a$ constant by reparametrization. There remain two translations (or conformal Killing vectors), a constant

[12]We count the number of real components here.

U(1), and constant Weyl transformation, the latter two eliminating two of the four degrees of freedom of $e_m{}^a$. In total we are left with two ordinary moduli, no odd moduli, and two translations as residual symmetries.

For odd-spin structure, D_m has zero modes on spinors, and we may set χ_m and $e_m{}^a$ only to a constant, but not necessarily to zero. There remain two translations and two constant supersymmetries (superconformal Killing spinors), a constant U(1) and Weyl transformation, and two constant super Weyl transformations as residual symmetries. The constant super Weyl transformations are eliminated by making the constant χ_m γ-traceless, and the U(1) and Weyl are used to restrict $e_m{}^a$ to two components. In total we are left with two moduli, two odd moduli, two translations, and two supersymmetries as residual symmetries.

To conclude, we obtain the decomposition

$$\{H_A{}^B\}=\{\delta\Sigma\}\oplus\{\delta L\}\oplus\{\text{Range}\mathcal{P}_1\}\oplus\{2\text{ moduli }\delta e_m{}^a\}\oplus\{\text{odd moduli}\}\ , \qquad (3.63)$$

where {odd moduli} is zero for even-spin structure and parametrized by γ-traceless, constant χ_m for odd-spin structure.

Early investigations of supermoduli parameters and their role in superstring perturbation theory are those of D'Hoker and Phong (1986b), Friedan, Martinec, and Shenker (1986), Moore, Nelson, and Polchinski (1986), and Chaudhuri, Kawai, and Tye (1987).

F. Null spaces of superderivatives and Laplacians

In this section we examine the structure of null spaces of superderivatives $\mathcal{D}^n_\pm$ and their associated Laplacians $\square_n^{(\pm)}$, as well as the relation between these null spaces. Questions relating to these issues have already come up in Secs. III.D and III.E with regard to the scalar Laplacian $\square_0$ and the Faddeev-Popov operator $\mathcal{P}_1^\dagger\mathcal{P}_1$ and will be essential to the analysis of the super Weyl and superholomorphic anomalies later on.

To gain insight into the behavior of Ker$\mathcal{D}^n_-$ and Ker$\square_n^{(-)}$, we note that the relation between the two kernels does not depend on super-reparametrizations or local U(1) transformations. Thus we may simplify the analysis by working in Wess-Zumino gauge and by choosing a slice for which χ_m is γ-traceless:

$$\chi_z{}^+=\chi_{\bar z}{}^-=0\ . \qquad (3.64)$$

We shall see that generically the relation between these kernels also does not depend on super Weyl rescalings. We introduce the field V of U(1) weight n, and its complex conjugate $\bar V$ of U(1) weight $-n$:

$$\begin{aligned} V&=V_0+\theta V_+ +\bar\theta V_- +i\theta\bar\theta V_1\ ,\\ \bar V&=\bar V_0+\theta\bar V_+ +\bar\theta\bar V_- +i\theta\bar\theta\bar V_1\ , \end{aligned} \qquad (3.65)$$

so that the U(1) weights of V_0, V_+, V_-, and V_1 are n, $n+\frac12$, $n-\frac12$, and n, respectively. (Or course these discrepancies arise because we choose Wess-Zumino gauge.) We easily find that

$$\begin{aligned} (\mathcal{D}^n_- V)=V_- &+\theta\left[-iV_1+\frac{i}{2}nAV_0\right]\\ &+\bar\theta(D_{\bar z}V_0+\tfrac12\chi_{\bar z}{}^+V_+)\\ &+\theta\bar\theta\left[\frac{i}{4}(2n+1)AV_- -\tfrac14\chi_{\bar z}{}^+\chi_z{}^-V_-\right.\\ &\left.\quad -\tfrac12\chi_{\bar z}{}^+D_zV_0-D_{\bar z}V_+ +in\Lambda_-V_0\right]\ . \end{aligned} \qquad (3.66)$$

To compute $\square_n^{(-)}V$, it is useful to evaluate

$$\begin{aligned} \int d\bar\theta\, d\theta\, E\mathcal{D}_+^{-n}\bar V\mathcal{D}^n_- V=e\Big[&D_z\bar V_0D_{\bar z}V_0-\bar V_+D_{\bar z}V_+ -\bar V_-D_zV_- +\tfrac12\chi_{\bar z}{}^+\bar V_+D_zV_0+\tfrac12\chi_{\bar z}{}^+D_z\bar V_0V_+ +\tfrac12\chi_z{}^-\bar V_-D_{\bar z}V_0\\ &+\tfrac12\chi_z{}^-D_{\bar z}\bar V_0V_- +\left[\bar V_1-\frac{n}{2}A\bar V_0\right]\left[V_1-\frac{n}{2}AV_0\right]+inA\bar V_+V_- +\tfrac14\chi_z{}^-\chi_{\bar z}{}^+\bar V_+V_-\\ &-\tfrac14\chi_z{}^-\chi_{\bar z}{}^+\bar V_-V_+ +in\bar V_+\Lambda_-V_0+in\bar V_0\Lambda_+V_-\Big]\ . \end{aligned} \qquad (3.67)$$

The vanishing of $\square_n^{(-)}V$ can then be gotten by variation with respect to $\bar V$ and one obtains

$$\begin{aligned} &D_zD_{\bar z}V_0+\frac{n}{2}A\left[V_1-\frac{n}{2}AV_0\right]-in\Lambda_+V_- +\tfrac12D_z(\chi_{\bar z}{}^+V_+)+\tfrac12D_{\bar z}(\chi_z{}^-V_-)=0\ ,\\ &D_zV_- +\tfrac12\chi_z{}^-D_{\bar z}V_0-\tfrac14\chi_z{}^-\chi_{\bar z}{}^+V_+=0\ ,\\ &D_{\bar z}V_+ +inAV_- +in\Lambda_-V_0+\tfrac12\chi_{\bar z}{}^+D_zV_0+\tfrac14\chi_z{}^-\chi_{\bar z}{}^+V_-=0\ ,\\ &V_1-\frac{n}{2}AV_0=0\ . \end{aligned} \qquad (3.68)$$

These equations are still rather formidable, and we shall take the following approach. We consider the case of zero gravitino field $\chi=0$ first, so that the equations reduce to

$$\begin{aligned} &D_z D_{\bar z} V_0=0 \quad \text{or } P_n^\dagger P_n V_0=0 , \\ &D_z V_- =0 \quad \text{or } P_{n-1/2}^\dagger V_- =0 , \\ &D_{\bar z} V_+ + in A V_- =0 \quad \text{or } P_{n-1/2} V_+ + in A V_- =0 , \\ &V_1 - \frac{n}{2} A V_0 =0 . \end{aligned} \tag{3.69}$$

These equations should now be compared with those obtained from $\mathcal{D}_-^n V=0$ in Eq. (3.66) at $\chi=0$, for which we find

$$\begin{aligned} &D_{\bar z} V_0=0 \quad \text{or } P_n V_0=0 , \\ &V_- =0 , \\ &D_{\bar z} V_+ =0 \quad \text{or } P_{n-1/2} V_+ =0 , \\ &V_1 - \frac{n}{2} A V_0 =0 . \end{aligned} \tag{3.70}$$

The first and the last equations of (3.69) and (3.70) are clearly equivalent.

Although the second and third equations in (3.69) and (3.70) seem different at first sight, we shall now show that, generically, they will also be equivalent. Indeed, let us derive an expression for the number of solutions $V_-\neq 0$ to (3.69). From $D_z^{n-1/2} V_- =0$, we have

$$V_- = \sum_{\alpha=1}^{N} p_\alpha \phi_\alpha ,$$

where ϕ_α span a basis for $\mathrm{Ker} D_z^{n-1/2}$, and N is its dimension. In order for the third equation of (3.69) to be consistent, AV_- must be in $\mathrm{Range} D_{\bar z}^{n+1/2}$, or equivalently it must be orthogonal to $\mathrm{Ker} D_z^{n-1/2}$. Thus the coefficients $p_\alpha \in \mathbf{C}$ must satisfy

$$\sum_{\alpha=1}^{N} \langle \phi_\beta | A \phi_\alpha \rangle p_\alpha =0 ,$$

$$\langle \phi_\beta | A \phi_\alpha \rangle = \int d^2 z \sqrt{g}\, \bar\phi_\beta A \phi_\alpha ,$$

and the number of nonzero solutions V_- to Eq. (3.69) must be $\#(V_-\neq 0)=\dim \mathrm{Ker} \langle \phi_\beta | A \phi_\alpha \rangle$. Generically, the matrix $\langle \phi_\beta | A\phi_\alpha\rangle$ will be nondegenerate and thus $\#(V_-\neq 0)=0$. For example, this will be the case when A is any positive function.

Thus, for $h\neq 1$ and $n\neq 0$, we have established the validity of $\mathrm{Ker}\mathcal{D}_-^n = \mathrm{Ker}\Box_n^{(-)}$ at least at the special point $\chi=0$. What happens when $\chi\neq 0$? In this case, we shall assume that χ results from a finite-dimensional space (parametrized by Grassmann-valued odd moduli) and we shall assume that χ is linear in these odd moduli ζ^a. Clearly, then, the different unknowns will be functions of ζ^a, but of course since there are a finite number of Grassmannian ζ's, these functions are just polynomials of bounded degree. It is not hard to see that one could expand

$$V_i = V_i^{(0)} + V_i^{(2)} + \cdots , \quad i=0,1,2,\ldots , \tag{3.71}$$

where the superscript denotes the degree of homogeneity in ζ. Now from the previous arguments when $\chi=0$, we know that for $h\geq 2$ and $n<0$ or $h=0$ and $n>0$, there are no solutions to order (0): $V_i^{(0)}=0$. But if this is so, the equation for $V_i^{(2)}$ is the same as for $V_i^{(0)}$, since all the perturbation terms are of order χ at least, and so on. One finds that V_i must identically vanish as soon as $V_i^{(0)}$ has to vanish.

We shall now discuss the above relations between null spaces for different genera. At this point, it is appropriate to deduce some generalizations that will prove fundamental later on. For $h\geq 2$ and A generic one has

$$\begin{aligned} &\mathrm{Ker}\Box_n^{(+)} = \mathrm{Ker}\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n = \mathrm{Ker}\mathcal{D}_+^n =0, \quad n\geq \tfrac{1}{2} , \\ &\mathrm{Ker}\Box_n^{(-)} = \mathrm{Ker}\mathcal{D}_+^{n-1/2}\mathcal{D}_-^n = \mathrm{Ker}\mathcal{D}_-^n =0, \quad n\leq -\tfrac{1}{2} . \end{aligned} \tag{3.72}$$

As for the kernel of the square of the Laplacian $\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n$ for $n\leq -1$,

$$\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n \mathcal{D}_-^{n+1/2}\mathcal{D}_+^n V=0 , \tag{3.73}$$

we can deduce using Eq. (3.72) that $\mathcal{D}_+^n \mathcal{D}_-^{n+1/2}\mathcal{D}_+^n V=0$, and with the help of Eq. (3.72) again, we find $\mathcal{D}_+^n V=0$, which implies that

$$\mathrm{Ker}(\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n)^2 \subset \mathrm{Ker}\mathcal{D}_+^n . \tag{3.74}$$

Since one manifestly also has the inclusion in the opposite sense, these kernels are in fact equal to one another, even though they need not be empty. Of course, one has an analogous statement for the other Laplacian. Putting these conclusions together, we have

$$\begin{aligned} &\mathrm{Ker}(\Box_n^{(+)})^2 = \mathrm{Ker}(\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n)^2 = \mathrm{Ker}\mathcal{D}_+^n, \quad n\leq -1 , \\ &\mathrm{Ker}(\Box_n^{(-)})^2 = \mathrm{Ker}(\mathcal{D}_+^{n-1/2}\mathcal{D}_-^n)^2 = \mathrm{Ker}\mathcal{D}_-^n, \quad n\geq 1 . \end{aligned} \tag{3.75}$$

In the case of the sphere $h=0$, the situation is precisely reversed. It is the $\mathcal{D}_-^n$ that have no zero modes for positive n, and it is readily established that

$$\begin{aligned} &\mathrm{Ker}\mathcal{D}_+^n =0, \quad n\leq -\tfrac{1}{2} , \\ &\mathrm{Ker}\mathcal{D}_-^n =0, \quad n\geq \tfrac{1}{2} , \end{aligned} \tag{3.76}$$

and similarly for their squares. By analogy with the higher-genus case, this implies the following identities between kernels of Laplacians:

$$\begin{aligned} &\mathrm{Ker}(\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n)^2 = \mathrm{Ker}\mathcal{D}_+^n, \quad n\geq \tfrac{1}{2} , \\ &\mathrm{Ker}(\mathcal{D}_+^{n-1/2}\mathcal{D}_-^n)^2 = \mathrm{Ker}\mathcal{D}_-^n, \quad n\leq -\tfrac{1}{2} . \end{aligned} \tag{3.77}$$

For the torus $h=1$, the nongeneric choice $A=0$ is natural from several points of view, as was already noted at the end of Sec. III.E. For $A=0$ and flat metric, a direct inspection shows that

$$\mathrm{Ker}(\Box_n^{(+)})^2 = \{ V=V_0+\theta V_+ + \bar\theta V_- + i\theta\bar\theta V_1 ;\ V_0, V_\pm, V_1 \text{ constants}\} ,$$

$$\mathrm{Ker}(\Box_n^{(+)}) = \mathrm{Ker}(\Box_n^{(+)})^2 \cap \{ V_1 =0\} ,$$

$$\mathrm{Ker}\mathcal{D}^n_+=\mathrm{Ker}(\Box_n^{(+)})\cap\{V_+=0\}\ .$$

On real fields, V_- equals V_+ and will vanish in the last case. For even-spin structure, all constant spinors vanish as well.

For $n=0$, the above arguments do not apply. The A term is absent in the third equation of (3.69), and at $\chi=0$, V_- is a Dirac zero mode. For odd-spin structure, there exists at least one such zero mode, and so $\mathrm{Ker}\Box_0^{(-)}\neq\mathrm{Ker}\mathcal{D}^0_-$. Whereas $\mathrm{Ker}\mathcal{D}^0_+$ reduces to constant superfields (when acting on real superfields), $\mathrm{Ker}\Box_0^{(+)}$ depends on moduli through the dependence of the number of Dirac zero modes on moduli, but may also depend on the odd moduli. However, the following argument will show that again $\mathrm{Ker}(\Box_0)^2=\mathrm{Ker}\Box_0$ generically. Consider the equation that must be satisfied by an element of $\mathrm{Ker}(\Box_0)^2$ not in $\mathrm{Ker}\Box_0$:

$$\Box_0 V=c+\theta\eta+\bar\theta\bar\eta \qquad (3.78)$$

with c constant and η a holomorphic spinor. For $\chi=0$, one readily finds that $\eta=0$, and integration over the surface must yield zero because $\Box_0$ is a derivative, so that

$$0=\int d^2\mathbf{z}\,E\Box_0 V=c\int d^2\mathbf{z}\,E\ . \qquad (3.79)$$

Now, generically, the area $\int d^2\mathbf{z}\,E$ will not vanish, though of course it need not be of definite sign. For constant-curvature geometries, indeed the area cannot vanish because of the Gauss-Bonnet formula for the Euler number of Eq. (3.38), when $h\neq 1$, and similarly the area will not vanish on any regular geometry. If that is so, then the constant must vanish and $V\in\mathrm{Ker}\Box_0$. We have thus established that

$$\mathrm{Ker}(\Box_0)^2=\mathrm{Ker}\Box_0\ . \qquad (3.80)$$

It will also be useful to simplify $\mathrm{Ker}(\Box_{1/2}^{(-)})^2$. Consider one of its elements V,

$$\mathcal{D}^0_+\mathcal{D}^{1/2}_-\mathcal{D}^0_+\mathcal{D}^{1/2}_-V=0\ . \qquad (3.81)$$

Multiplying to the left by $\mathcal{D}^{1/2}_-$ and using Eq. (3.80), we get

$$\mathcal{D}^{1/2}_-\mathcal{D}^0_+\mathcal{D}^{1/2}_-V=0\ . \qquad (3.82)$$

The spurious solutions satisfy

$$\mathcal{D}^{1/2}_-V=\mathrm{const}\ , \qquad (3.83)$$

and upon integrating over the supersurface, as in Eq. (3.79), we find again that $\mathcal{D}^{1/2}_-V=0$. Hence we conclude

$$\mathrm{Ker}(\Box_{1/2}^{(-)})^2=\mathrm{Ker}\mathcal{D}^{1/2}_-\ . \qquad (3.84)$$

The nongeneric slices are always easily treated as limits of generic slices.

G. Supermoduli space and its complex structure

In Sec. III.E, we identified the infinitesimal changes in the supergeometry of a super Riemann surface with Teichmüller deformations, spanning $\mathrm{Ker}\mathcal{P}_1^\dagger$. The space of supergeometries of genus h, satisfying the torsion constraints (3.11) inequivalent under the symmetry groups $\mathrm{sDiff}_0(M)$, $\mathrm{sWeyl}(M)$, and $\mathrm{sU}(1)$ is super Teichmüller space

$$s\mathcal{T}_h=\frac{\{E_M{}^A,\Omega_M \text{ satisfying } (3.11)\}}{\{\mathrm{sDiff}_0(M)\times\mathrm{sWeyl}(M)\times\mathrm{sU}(1)\}}\ . \qquad (3.85)$$

The quotient of the full super-reparametrization group $\mathrm{sDiff}(M)$ by $\mathrm{sDiff}_0(M)$ is the ordinary mapping class group MCG_h (acting on surface with spin structures) so that we may define *supermoduli* space as

$$\begin{aligned} s\mathcal{M}_h&=s\mathcal{T}_h/\mathrm{MCG}_h\ ,\\ \mathrm{MCG}_h&=\mathrm{sDiff}(M)/\mathrm{sDiff}_0(M)\\ &=\mathrm{Diff}(M)/\mathrm{Diff}_0(M)\ . \end{aligned} \qquad (3.86)$$

The complex nature of $s\mathcal{M}_h$ can be seen by viewing it as the space of *superconformal classes*. Indeed, recall that the complex structure on a super Riemann surface $J_M{}^N$, introduced in Eq. (3.23), is unchanged under $\mathrm{sWeyl}(M)$ and $\mathrm{sU}(1)$ and that it is a tensor under $\mathrm{sDiff}(M)$. Thus we have

$$s\mathcal{M}_h=\{J_M{}^N\}/\mathrm{sDiff}(M)\ , \qquad (3.87)$$

where $J_M{}^N J_N{}^P=-\delta_M{}^P$, and it is understood that $J_M{}^N$ arises from a supergeometry satisfying the torsion constraints (3.11). There are now natural holomorphic coordinates on $s\mathcal{M}_h$, as can be seen by exhibiting a natural complex structure on it. The tangent space at $J_M{}^N$ can be identified with

$$T(s\mathcal{M}_h)=\{J_M{}^N;J_M{}^N\delta J_N{}^P+\delta J_M{}^N J_N{}^P=0\}\ , \qquad (3.88)$$

on which there is a natural map

$$\mathcal{J}:\ T(s\mathcal{M}_h)\to T(s\mathcal{M}_h),\quad \mathcal{J}(\delta J_M{}^N)=J_M{}^P\delta J_P{}^N \qquad (3.89)$$

whose square is minus the identity

$$\mathcal{J}^2(\delta J_M{}^N)=\mathcal{J}(J_M{}^P\delta J_P{}^N)=-\delta J_M{}^N\ . \qquad (3.90)$$

Thus $\mathcal{J}$ is an almost complex structure on $s\mathcal{M}_h$. It is actually integrable, as can be seen by considering the following one-forms:

$$\begin{aligned} \Gamma_M{}^N&=dJ_M{}^N-i\mathcal{J}(dJ_M{}^N)\ ,\\ \bar\Gamma_M{}^N&=dJ_M{}^N+i\mathcal{J}(dJ_M{}^N)\ . \end{aligned} \qquad (3.91)$$

The exterior derivatives are easily obtained,

$$d\Gamma_M{}^N=\frac{i}{4}(\Gamma_M{}^P\wedge\bar\Gamma_P{}^N+\bar\Gamma_M{}^P\wedge\Gamma_P{}^N)\ , \qquad (3.92)$$

and the almost complex structure is integrable provided $d\Gamma$ vanishes where $\Gamma=0$, which is obviously the case. Notice that this integrability condition uses only the fact that $J_M{}^N$ itself is a complex structure on the super Riemann surface; it does not further depend on the torsion constraints. One concludes that $s\mathcal{M}_h$ is a supercomplex ($V-$) manifold.

A perhaps more concrete description of supermoduli space may be given in terms of constant-(super)curvature geometries. The key step in analogy with the bosonic case is the choice of a slice for sWeyl(M) that generalizes constant curvature. The correct choice is

$$R_{+-}=\text{const} . \tag{3.93}$$

This slice is clearly invariant under super-reparametrizations and local U(1) transformations. it also has the advantage of implying that all components of the torsion and curvature are constant, as one can readily deduce from Eqs. (3.11)–(3.13). A simple interpretation of Eq. (3.93) can be obtained in Wess-Zumino gauge. Recall that in this gauge R_{+-} expanded in powers of θ is given by Eq. (3.36), so that A is constant, $\Lambda_\alpha=0$, and $C=0$. Finally we can also argue that (3.93) is indeed a slice, in the sense that any supergeometry $E_M{}^A$ can be brought back to a supergeometry $\hat{E}_M{}^A$ satisfying Eq. (3.93) by a unique super Weyl transformation. In fact Eq. (3.20) shows that the parameter Σ of the transformation must satisfy a super Liouville equation,

$$2i\mathcal{D}_+\mathcal{D}_-\Sigma+R_{+-}-e^{\Sigma}\hat{R}_{+-}=0 . \tag{3.94}$$

This equation is locally soluble, and there is no topological obstruction besides the Euler characteristic.

By restricting ourselves to the gauge slice $R_{+-}=\text{const}$, we have eliminated the action of the super Weyl group. To factor out the remaining symmetries we simply pass to cosets. More precisely, consider $dz^M\Omega_M$ as living in the space of one-forms modulo exact forms, and set $s\mathcal{M}_{\text{const}}$ to be the space of constant R_{+-} supergeometries modulo all local U(1) transformations. We can now define supermoduli space as the coset space

$$s\mathcal{M}_h=s\mathcal{M}_{\text{const}}/\text{sDiff}(M) . \tag{3.95}$$

From the orthogonal decomposition of $\{H_A{}^B\}$ given in Eq. (3.53), it is evident that $s\mathcal{M}_h$ is a supermanifold whose tangent space at each supergeometry is

$$T(s\mathcal{M}_h)=\text{Ker}\mathcal{P}_1^\dagger , \tag{3.96}$$

so that its dimension is also given by Eq. (3.58) for $h\neq 1$, whereas for $h=1$, the tangent space is $\{2 \text{ even moduli } e_m{}^a\}\oplus\{\text{odd moduli}\}$.

The holomorphic structure $\mathcal{J}$ on supermoduli space and its integrability are due to D'Hoker and Phong (1987a).

H. Determinants, super Weyl and local U(1) anomalies

In order to reduce the string path integrals over supergeometries to integrals over supermoduli space, one needs the behavior of the superdeterminants of the covariant derivatives with respect to super Weyl and local U(1) transformations. We start by considering the Laplacians $\Box_n^{(+)}$ and $\Box_n^{(-)}$ of Eq. (3.9).

The local part of the super Weyl anomaly has been evaluated by Martinec (1983). The zero modes for superdeterminants are, however, a nontrivial issue, since the nonpositivity of the norms could cause the kernels of $(\mathcal{D}_+\mathcal{D}_-)^2$, $\mathcal{D}_+\mathcal{D}_-$, and $\mathcal{D}_-$ to be distinct (cf. Sec. III.F). Each of these spaces has its own transformation law under super Weyl scalings, so it is important to determine which one will combine with $\text{sdet}(\mathcal{D}_+\mathcal{D}_-)^2$ to produce a *local* anomaly. Another consequence of the nonpositivity of the norms is that the Laplacians $\Box_n$ need not be diagonalizable. In addition, even though they are the product of an operator times its adjoint, they need not be positive. In fact, writing $\Box_n$ in components, it is clear that besides the standard Laplacians acting on ordinary functions, there is also a piece behaving like a first-order differential operator, so that the spectrum in general extends from $-\infty$ to $+\infty$. The square of $\Box_n$ is still not a positive operator in general, but is at least bounded from below. Strictly speaking, the last property has been shown only on surfaces of constant curvature by Aoki (1988), but is is clear that a continuous super Weyl transformation may alter the lower bound, but will not send it to $-\infty$. The heat kernel $\exp[-t(\Box_n^{(\pm)})^2]$ may thus tend to infinity as $t\to\infty$ in an exponential fashion, and ζ-function regularization cannot be applied to define the corresponding superdeterminants. We now provide a detailed analysis of these issues. We define the superdeterminant through an exponential regulator, depending on a complex parameter s,

$$\begin{aligned}\ln\delta_n^{(\pm)}(s)&=\ln\text{sdet}[(\Box_n^{(\pm)})^2+s]\\&=-\int_\varepsilon^\infty\frac{dt}{t}e^{-ts}\text{sTr}e^{-t(\Box_n^{(\pm)})^2} ,\end{aligned} \tag{3.97}$$

which converges absolutely for Re(s) sufficiently large and $\varepsilon>0$. Throughout the complex s plane, $\delta_n^{(\pm)}(s)$ is defined by analytic continuation. For constant-curvature supergeometries $\delta_n^{(\pm)}$ is meromorphic throughout **C**, and this is enough to argue that $\delta_n^{(\pm)}$ will be meromorphic for any supergeometry, as will become clear through the super Weyl anomaly calculation. Thus, around $s=0$, $\delta_n^{(\pm)}$ will in general have the following behavior:

$$\delta_n^{(\pm)}(s)=s^{N_n^\pm}\text{sdet}'(\Box_n^{(\pm)})^2+O(s^{N_n^\pm+1}) , \tag{3.98}$$

where $N_n^\pm$ are positive or negative integers, formally corresponding to the difference between the number of even zero modes and odd zero modes. This relation defines the superdeterminant of $(\Box_n^{(\pm)})^2$, whereas the superdeterminant of $\Box_n^{(\pm)}$ itself is the square root

$$\text{sdet}'(\Box_n^{(\pm)})^2\equiv(\text{sdet}'\Box_n^{(\pm)})^2 . \tag{3.99}$$

To examine the behavior under super Weyl transformations of the determinants in Eq. (3.99), we determine the super Weyl change of $\delta_n^{(\pm)}$ and analytically continue to $s=0$. We shall restrict attention to $\Box_n^{(+)}$ and quote the results for $\square_n^{(-)}$ at the end:

$$\delta \ln \delta_n^{(+)}(s) = 2\int_\varepsilon^\infty dt\, e^{-ts} \times \mathrm{sTr}(\delta\square_n^{(+)}\square_n^{(+)} e^{-t(\square_n^{(+)})^2}) . \quad (3.100)$$

The changes of the superderivatives are given by

$$\delta\mathcal{D}_+^n = (n-\tfrac{1}{2})\delta\Sigma\mathcal{D}_+^n - n\mathcal{D}_+^n\delta\Sigma ,$$
$$\delta\mathcal{D}_-^n = -(n+\tfrac{1}{2})\delta\Sigma\mathcal{D}_-^n + n\mathcal{D}_-^n\delta\Sigma , \quad (3.101)$$
$$\delta\square_n^{(\pm)} = (-1\mp n)\delta\Sigma\square_n^{(\pm)} \pm 2n\mathcal{D}_\mp^{n\pm 1/2}\delta\Sigma\mathcal{D}_\pm^n \mp n\square_n^{(\pm)}\delta\Sigma ,$$

so that[13]

$$\begin{aligned}\mathrm{sTr}(\delta\square_n^{(+)}\square_n^{(+)}e^{-t(\square_n^{(+)})^2}) &= -(2n+1)\mathrm{sTr}\delta\Sigma(\square_n^{(+)})^2 e^{-t(\square_n^{(+)})^2} - 2n\,\mathrm{sTr}\delta\Sigma(\square_{n+1/2}^{(-)})^2 e^{-t(\square_{n+1/2}^{(-)})^2} \\ &= (2n+1)\frac{\partial}{\partial t}\mathrm{sTr}\delta\Sigma e^{-t(\square_n^{(+)})^2} + 2n\frac{\partial}{\partial t}\mathrm{sTr}\delta\Sigma e^{-t(\square_{n+1/2}^{(-)})^2} \end{aligned} \quad (3.102)$$

Inserting this result into Eq. (3.100), one finds

$$\delta \ln \delta_n^{(+)} = 2\int_\varepsilon^\infty dt\, e^{-ts}\left[(2n+1)\frac{\partial}{\partial t}\mathrm{sTr}\delta\Sigma e^{-t(\square_n^{(+)})^2} + 2n\frac{\partial}{\partial t}\mathrm{sTr}\delta\Sigma e^{-t(\square_{n+1/2}^{(-)})^2}\right] \quad (3.103)$$

Integrating by parts yields

$$\begin{aligned}\delta \ln \delta_n^{(+)} &= 2e^{-ts}[(2n+1)\mathrm{sTr}\delta\Sigma e^{-t(\square_n^{(+)})^2} + 2n\,\mathrm{sTr}\delta\Sigma e^{-t(\square_{n+1/2}^{(-)})^2}]\,\Big|_\varepsilon^\infty \\ &\quad + 2s\int_\varepsilon^\infty dt\, e^{-ts}[(2n+1)\mathrm{sTr}\delta\Sigma e^{-t(\square_n^{(+)})^2} + 2n\,s\mathrm{Tr}\delta\Sigma e^{-t(\square_{n+1/2}^{(-)})^2}] . \end{aligned} \quad (3.104)$$

Since the expression is defined for $\mathrm{Re}(s)$ sufficiently large, the contribution from infinity in the first term cancels out, and the remaining traces of heat kernels are well defined at $s=0$. In the second term, the only nonzero contribution can arise in the limit where $s\to 0$ if the integral produces a simple pole at $s=0$. To see whether this happens, we remark that the general form of the contribution to the supertraces is $t^p e^{-\lambda t}$, where p and λ are arbitrary and independent of s. Substituted into the integral in (3.104), we find that such a contribution produces

$$s\int_\varepsilon^\infty dt\, e^{-ts}t^p e^{-\lambda t} = \frac{s}{(s+\lambda)^{p+1}}\Gamma(p+1) + O(\varepsilon) . \quad (3.105)$$

One notices that, whatever the value of p, a nonzero result is produced as $s\to 0$ only if $\lambda = 0$. In the trace of the superheat kernel, this results from the zero modes, so that $p=0$ as well. Collecting these results, we get

$$\begin{aligned}\delta \ln \mathrm{sdet}'\square_n^{(+)} &= \tfrac{1}{2}\lim_{s\to 0}\ln\delta_n^{(+)}(s) \\ &= -(2n+1)\mathrm{sTr}\delta\Sigma e^{-\varepsilon(\square_n^{(+)})^2} - 2n\,\mathrm{sTr}\delta\Sigma e^{-\varepsilon(\square_{n+1/2}^{(-)})^2} + (2n+1)\mathrm{sTr}\delta\Sigma\,|_{\mathrm{Ker}(\square_n^{(+)})^2} + 2n\,\mathrm{sTr}\delta\Sigma\,|_{\mathrm{Ker}(\square_{n+1/2}^{(-)})^2} . \end{aligned} \quad (3.106)$$

The terms involving ε on the right-hand side of Eq. (3.106) are local functions of $\delta\Sigma$ in the limit where $\varepsilon\to 0$, and their expressions can be gotten from a short-time expansion of the super heat kernel, which is derived in Appendix C:

$$\mathrm{sTr}\delta\Sigma e^{-\varepsilon(\square_n^{(+)})^2} = -i\frac{1+2n}{4\pi}(-)^{2n}\int d^2\mathbf{z}\,ER_{+-}\delta\Sigma + O(\varepsilon) , \quad (3.107)$$

$$\mathrm{sTr}\delta\Sigma e^{-\varepsilon(\square_n^{(-)})^2} = +i\frac{1-2n}{4\pi}(-)^{2n}\int d^2\mathbf{z}\,ER_{+-}\delta\Sigma + O(\varepsilon) .$$

Notice that, due to worldsheet supersymmetry, there is no term behaving like $1/\varepsilon$, as we had in the case of the bosonic string.

The traces of $\delta\Sigma$ restricted to the kernels of zero modes are familiar from the bosonic case, but much more care is needed for the case of the superstring, due to the fact that the kernel of the square of an operator may be different from the kernel of the operator itself, as we have seen in Sec. III.F.

For $h\geq 2$ and $n\geq\frac{1}{2}$, we have $\mathrm{Ker}(\square_n^{(+)})^2=0$ according to Eq. (3.72), and that $\mathrm{Ker}(\square_{n+1/2}^{(-)})^2=\mathrm{Ker}\mathcal{D}_-^{n+1/2}$ according to Eq. (3.75). The remaining trace can be linked to the change in the finite-dimensional determinant of elements[14] $\Phi_J\in\mathrm{Ker}\mathcal{D}_-^{n+1/2}$, using the fact that they scale as

[13]Note that the analogous calculation could have been performed using local U(1) transformations of the superderivatives. At this point one would have found that the contributions cancel and that the determinants are invariant.

[14]Henceforth J,K stand for mixed indices $J=(j,a),K=(k,b)$, etc., where $j=1,\ldots,3h-3$ and $a=1,\ldots,2h-2$.

$\Phi_J = e^{-(n+1/2)\Sigma}\hat{\Phi}_J$:

$$\delta \ln \mathrm{sdet}\langle \Phi_J \mid \Phi_K \rangle = \mathrm{sTr}\delta\langle \Phi_J \mid \Phi_K \rangle = -2n \sum_J \langle \Phi_J \mid \delta\Sigma \mid \Phi_J \rangle \ . \tag{3.108}$$

Putting these together, we find (for $h \geq 2$ and $n \geq \frac{1}{2}$)

$$\delta \ln \frac{\mathrm{sdet}\square_n^{(+)}}{\mathrm{sdet}\langle \Phi_J \mid \Phi_K \rangle} = i\frac{4n+1}{4\pi}(-)^{2n}\int d^2\mathbf{z}\, ER_{+-}\delta\Sigma \ . \tag{3.109}$$

It is straightforward to see that the same arguments apply for $h=0$, provided that $n \leq -\frac{1}{2}$.

For $h \geq 2$ and $n \leq -1$, exactly the opposite situation is produced, and we have according to Eq. (3.72) that $\mathrm{Ker}(\square_{n+1/2}^{(-)})^2 = 0$ and $\mathrm{Ker}(\square_n^{(+)})^2 = \mathrm{Ker}\mathcal{D}_+^n$. The remaining trace can now be linked to the change in the finite-dimensional determinant of elements $\Psi_\alpha \in \mathrm{Ker}\mathcal{D}_+^n$, which scale as $\Psi_\alpha = e^{n\Sigma}\hat{\Psi}_\alpha$. Thus

$$\delta \ln \mathrm{sdet}\langle \Psi_\alpha \mid \Psi_\beta \rangle = \mathrm{sTr}\delta\langle \Psi_\alpha \mid \Psi_\beta \rangle = -(2n+1)\sum_\alpha \langle \Psi_\alpha \mid \delta\Sigma \mid \Psi_\alpha \rangle \ , \tag{3.110}$$

and putting all together, we find

$$\delta \ln \frac{\mathrm{sdet}\square_n^{(+)}}{\mathrm{sdet}\langle \Psi_\alpha \mid \Psi_\beta \rangle} = i\frac{4n+1}{4\pi}(-)^{2n}\int d^2\mathbf{z}\, ER_{+-}\delta\Sigma \ . \tag{3.111}$$

Similarly for the sphere, this formula will hold for $n \geq \frac{1}{2}$.

The cases $n=0$ and $n=-\frac{1}{2}$ are symmetrical, so we shall limit our discussion to $n=0$. The novelty here is that one of the finite-dimensional traces is absent from Eq. (3.106), the other one being taken over $\mathrm{Ker}(\square_0)^2$. Though $\mathrm{Ker}(\square_0)^2$ might be larger than $\mathrm{Ker}\square_0$, it was argued in Eq. (3.80) that this is not the generic case. Since the zero modes of $\square_0$ are super Weyl invariant, we readily deduce that Eq. (3.111) holds, but now with $\Psi_\alpha \in \mathrm{Ker}\square_0$, which may be larger than $\mathrm{Ker}\mathcal{D}_+^0$. Similarly, since $\Phi_J \in \mathrm{Ker}\mathcal{D}_-^{1/2}$ scales as $\Phi_J = e^{-\Sigma/2}\hat{\Phi}_J$, it is clear that $\mathrm{sdet}\langle \Phi_J \mid \Phi_K \rangle$ is super Weyl invariant, in analogy with the finite-dimensional determinant over inner products of holomorphic Abelian differentials in the bosonic string. We might be tempted to call the $\frac{1}{2}$ differentials $\Phi_J \in \mathrm{Ker}\mathcal{D}_-^{1/2}$ *holomorphic super Abelian differentials.*

We may now collect all the above results for $h \geq 2$ or $h=0$ in one formula, and also integrate the infinitesimal $\delta\Sigma$'s to finite super Weyl transformations:

$$\ln \frac{\mathrm{sdet}'\square_n^{(+)}}{\mathrm{sdet}\langle \Phi_J \mid \Phi_K \rangle \mathrm{sdet}\langle \Psi_\alpha \mid \Psi_\beta \rangle} = \ln \frac{\mathrm{sdet}\hat{\square}_n^{(+)}}{\mathrm{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K \rangle \mathrm{sdet}\langle \hat{\Psi}_\alpha \mid \hat{\Psi}_\beta \rangle} - (1+4n)S_{sL}(\Sigma) \ , \tag{3.112}$$

where the local super Weyl anomaly is given by

$$S_{sL}(\Sigma) = \frac{1}{4\pi}\int d^2\mathbf{z}\, \hat{E}(\hat{\mathcal{D}}_+\Sigma\hat{\mathcal{D}}_-\Sigma - i\hat{R}_{+-}\Sigma) \tag{3.113}$$

and $\Phi_J \in \mathrm{Ker}\mathcal{D}_-^{n+1/2}$, except when $n=-\frac{1}{2}$, where $\Phi_J \in \mathrm{Ker}\square_0^{(-)}$ and $\Psi_\alpha \in \mathrm{Ker}\mathcal{D}_+^n$, except for $n=0$, where $\Psi_\alpha \in \mathrm{Ker}\square_0^{(+)}$. Similarly, we can derive the super Weyl anomaly for $\square_n^{(-)}$ and find

$$\ln \frac{\mathrm{sdet}'\square_n^{(-)}}{\mathrm{sdet}\langle \Phi_J \mid \Phi_K \rangle \mathrm{sdet}\langle \Psi_\alpha \mid \Psi_\beta \rangle} = \ln \frac{\mathrm{sdet}\hat{\square}_n^{(-)}}{\mathrm{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K \rangle \mathrm{sdet}\langle \hat{\Psi}_\alpha \mid \hat{\Psi}_\beta \rangle} - (1-4n)S_{sL}(\Sigma) \ . \tag{3.114}$$

Here $\Phi_J \in \mathrm{Ker}\mathcal{D}_-^n$, except for $n=0$, where it belongs to $\mathrm{Ker}\square_0^{(-)}$ and $\Psi_\alpha \in \mathrm{Ker}\mathcal{D}_+^{n-1/2}$, except for $n=\frac{1}{2}$, where it belongs to $\mathrm{Ker}\square_0^{(+)}$.

For the torus and a generic slice, it is clear that Eqs. (3.112)–(3.114) hold as well. If, on the other hand, the nongeneric slice $A=0$ is chosen, one should rather consider $\mathrm{sdet}'(\square_n^{(\pm)})^2$ and divide by the determinants of inner products of $\mathrm{Ker}(\square_n^{(\pm)})^2$.

I. Amplitudes as integrals over supermoduli

With the above analysis of the space of supergeometries, it is now easy to carry out the $DE_M{}^A$ integral. We shall limit ourselves to the case $h \geq 2$ and treat the sphere and the torus in Secs. III.L and III.M. In parallel with the bosonic case, we introduce a slice S of dimension $(6h-6, 4h-4)$, transversal to the action of $\mathrm{sDiff}_0(M)$ within the space of supergeometries. We parametrize the space of supergeometries by

$$E_M{}^A = e^V e^\Sigma e^{iL}\hat{E}_M{}^A \ , \tag{3.115}$$

with $\hat{E}_M{}^A$ in S and the exponentials representing the actions of the various symmetry groups. If m_J are coordinates for the slice S, $\hat{F}_J$ are the corresponding coordinate vectors in $T(s\mathcal{M}_{\mathrm{const}})$, and Φ_J is a basis for $\mathrm{Ker}\mathcal{P}_1^\dagger$, then the measure is obtained from the calculation of the Jacobian factor associated with the change of variables from $E_M{}^A$ to Σ, L, V, and m_J. With the orthogonal decomposition of Eq. (3.53), this Jacobian can easily be worked out, and we find

$$DE_M{}^A = (\mathrm{sdet}\mathcal{P}_1^\dagger + \mathcal{P}_1)^{1/2}\frac{\mathrm{sdet}\langle e^{iL}e^{-\Sigma/2}\hat{F}_J \mid \Phi_K \rangle_E^2}{\mathrm{sdet}\langle \Phi_J \mid \Phi_K \rangle_E} \times D\Sigma DL DV^M \prod dm_J \ . \tag{3.116}$$

The subscript in the inner product indicates which superzweibein is used in the pairing of tensors.

The super Weyl dependence of the various ingredients of Eq. (3.116) may be calculated in analogy with the bosonic string case. First one uses the fact that

$$\mathrm{Ker}\mathcal{P}_1^\dagger = \mathrm{Ker}\mathcal{D}_+^1 \oplus \mathrm{Ker}\mathcal{D}_-^{-1} \tag{3.117}$$

and that if $\Phi_J \in \mathrm{Ker}\mathcal{D}_\pm^{\pm 1}$ then $\hat{\Phi}_J = e^{(3/2)\Sigma}\Phi_J \in \mathrm{Ker}\hat{\mathcal{D}}_\pm^{\pm 1}$. Similar properties are easily derived for the local U(1) transformations using Eqs. (3.15)–(3.17). As a result, one finds that

$$\mathrm{sdet}\langle e^{iL} e^{-\Sigma/2}\hat{F}_J \mid \Phi_K\rangle_E = \mathrm{sdet}\langle \mu_J \mid \hat{\Phi}_K\rangle \ , \tag{3.118}$$

where the inner product on the right-hand side is now evaluated with respect to the supergeometry $\hat{E}_M{}^A$. The μ_J are dual super Beltrami differentials, in the sense of Sec. III.J below. Their bosonic analog appeared in Secs. II.E and II.G. Next, we recall from Sec. III.H the super Weyl scalings relevant to superstring theory:

$$\begin{aligned}\frac{\mathrm{sdet}\mathcal{P}_1^\dagger\mathcal{P}_1}{\mathrm{sdet}\langle \Phi_J \mid \Phi_K\rangle} &= \frac{\mathrm{sdet}\hat{\mathcal{P}}_1^\dagger + \hat{\mathcal{P}}_1}{\mathrm{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K\rangle} e^{-10 S_{sL}(\Sigma)} \ , \\ \frac{\mathrm{sdet}'\Box_0}{\mathrm{sdet}\langle \Psi_\alpha \mid \Psi_\beta\rangle} &= \frac{\mathrm{sdet}'\hat{\Box}_0}{\mathrm{sdet}\langle \hat{\Psi}_\alpha \mid \hat{\Psi}_\beta\rangle} e^{-S_{sL}(\Sigma)} \ ,\end{aligned} \tag{3.119}$$

where $\Psi_\alpha \in \mathrm{Ker}\Box_0$.

The first and second equations show that the nonlocal Σ dependence cancels out of Eq. (3.116). The local dependence on the super Weyl scaling Σ is canceled out by putting contributions of the Faddeev-Popov and matter determinants together, provided the dimension of space-time is $d=10$. Since we are dealing with the type-II string here, a potential local U(1) anomaly is canceled between left- and right-movers on the worldsheet. For the heterotic string, the absence of the local U(1) anomaly will put further constraints on the theory, which will be explained in Sec. III.N and amount to requiring the gauge group to have rank 16. Vertex operators will be determined so that the above symmetries of the measure are preserved, after all anomalous contributions have been taken into account.

Since the combined measure will be invariant under super-reparametrizations, local U(1), and super Weyl transformations, it really runs over the coset space of all $N=1$ supergeometries by these symmetries. The remaining coset space coincides precisely with that of all supercomplex structures, and was termed supermoduli space in Sec. III.G. Thus the domain of integration will be supermoduli space. The measure becomes

$$DE_M{}^A = (\mathrm{sdet}\hat{\mathcal{P}}_1^\dagger\hat{\mathcal{P}}_1)^{1/2} e^{-5S_{sL}(\Sigma)} D\Sigma DL DV^M \times \frac{\mathrm{sdet}\langle \mu_J \mid \hat{\Phi}_K\rangle}{\mathrm{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K\rangle} \prod_J dm_J \ . \tag{3.120}$$

Now the last equation in (3.119) shows that in the critical dimension $d=10$ the local super Weyl anomaly $S_{sL}(\Sigma)$ disappears as well, to yield the formula

$$Z_h = \Omega \int_{s\mathcal{M}_h} \prod_J dm_J \frac{\mathrm{sdet}\langle \mu_J \mid \hat{\Phi}_K\rangle}{\mathrm{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K\rangle} \left[\frac{8\pi^2 \mathrm{sdet}'\hat{\Box}_0}{\int d^2\mathbf{z}\, E}\right]^{-5} \times (\mathrm{sdet}\hat{\mathcal{P}}_1^\dagger\hat{\mathcal{P}}_1)^{1/2} \ . \tag{3.121}$$

As in the bosonic string, if we choose a slice within $s\mathcal{M}_{\mathrm{const}}$, this measure is manifestly a coset measure on $s\mathcal{M}_h$, which can be termed the super Weil-Petersson measure,

$$d(\mathrm{sWP}) = \frac{\mathrm{sdet}\langle \mu_J \mid \hat{\Phi}_K\rangle}{\mathrm{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K\rangle} \prod_J dm_J \ . \tag{3.122}$$

We shall often refer to the right-hand side of Eq. (3.122) as the super Weil-Petersson measure, even when the slice does not lie within $s\mathcal{M}_{\mathrm{const}}$. Such slices, e.g., those that depend holomorphically on supermoduli parameters, will be important later.

We conclude this section by noting that on-shell scattering amplitudes may be reduced in the same way to integrals over supermoduli by insertion of the proper vertex operators, as discussed in Sec. VIII. For the case of bosonic vertex operator insertions, one finds in general

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_h = \int_{s\mathcal{M}_h} \prod_J dm_J \frac{\mathrm{sdet}\langle \mu_J \mid \hat{\Phi}_K\rangle}{\mathrm{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K\rangle} \left[\frac{8\pi^2\mathrm{sdet}'\hat{\Box}_0}{\mathrm{sdet}\langle \hat{\Psi}_\alpha \mid \hat{\Psi}_\beta\rangle}\right]^{-5} (\mathrm{sdet}\hat{\mathcal{P}}_1^\dagger\hat{\mathcal{P}}_1)^{1/2} \langle\langle V_1(k_1)\cdots V_n(k_n)\rangle\rangle_{\hat{E}} \ , \tag{3.123}$$

where $\langle\langle\ \rangle\rangle$ stands for the fact that only the X^μ integral has been carried out (including the integration over all X^μ zero modes).

J. Formulation with superghosts

In this section the Faddeev-Popov determinant, together with the finite-dimensional determinants involving super Beltrami and superquadratic differentials, is recast in terms of a functional integral over superghost fields, and a local action is obtained on the worldsheet. The goal ultimately is to derive a formulation in Wess-Zumino gauge closely related to that of conformal field theory. This will be fully achieved in the next section, III.K.

Before deriving the superghost expression, we need a better insight into the nature of superquadratic and super Beltrami differentials.

1. Superquadratic and super Beltrami differentials

Holomorphic superquadratic differentials Φ_J are U(1) tensors of weight $\frac{3}{2}$ and are solutions to

$$\mathcal{D}_-^{3/2}\Phi_J = 0 \tag{3.124}$$

Recall that in Wess-Zumino gauge $\Phi_J=\phi_J^0+\theta\phi_J^1+(3i/4)A\phi_J^0\theta\bar\theta$ satisfies the equation

$$D_{\bar z}\phi_J^0+\tfrac{1}{2}\chi_{\bar z}^+\phi_J^1=0\ ,$$
$$D_{\bar z}\phi_J^1+\tfrac{1}{2}\chi_{\bar z}^+D_z\phi_J^0-\tfrac{3}{2}\Lambda_-\phi_J^0=0\ . \qquad (3.125)$$

When $\chi=0$, ϕ_J^0 and ϕ_J^1 are holomorphic $\frac{3}{2}$ and quadratic differentials, respectively, so in that particular case we may set $\phi_j^0=\phi_a^1=0$. The remaining components ϕ_a^0 and ϕ_j^1 are then the standard holomorphic differentials, and their number is in accord with index calculations. They are also naturally even Grassmann-valued elements. Away from $\chi=0$, the same number of solutions to Eq. (3.125) exists, and here ϕ_a^0 and ϕ_j^1 are even, whereas ϕ_j^0 and ϕ_1^1 are odd Grassmann elements. Putting all together, we have $5h-5$ holomorphic superquadratic differentials Φ_J, $3h-3$ of which are odd (Φ_j) and $2h-2$ of which are even (Φ_a).

Super Beltrami differentials μ_K with $K=(k,b)$ are dual to holomorphic super quadratic differentials and may be normalized as

$$\langle\mu_K\mid\Phi_J\rangle=\delta_{KJ}\ , \qquad (3.126)$$

so that there are again $5h-5$ μ_K's, $3h-3$ of which are odd (μ_k) and $2h-2$ of which are even (μ_a). More generally, super Beltrami differentials may also be viewed as inequivalent small deformations of the supergeometry of a super Riemann surface, belonging to the tangent space to supermoduli $T(s\mathcal{M}_h)$. (See the analogous discussion for the bosonic case in Secs. II.D and II.E). It will be convenient to introduce coordinates m_K for supermoduli space; m_k should be thought of as ordinary even moduli and m_b as odd moduli. The small deformations inequivalent under U(1), super Weyl, and super-reparametrizations could be parametrized by the component $H_-{}^z$ (and its complex conjugate) according to Eq. (3.45). Thus the super Beltrami differentials μ_K may naturally be defined as

$$\mu_K=(H_-{}^z)_K=E_-{}^M\frac{\partial E_M{}^z}{\partial m_K}\ . \qquad (3.127)$$

It follows that super Beltrami differentials satisfy the integrability condition given by

$$\frac{\partial\mu_K}{\partial m_L}+(-)^{KL}\frac{\partial\mu_L}{\partial m_K}=0\ . \qquad (3.128)$$

It is instructive to look at this structure in Wess-Zumino gauge, where we have

$$H_-{}^z=\bar\theta(e_{\bar z}{}^m\delta e_m{}^z-\theta\delta\chi_{\bar z}{}^+)\ . \qquad (3.129)$$

There is also a contribution from a Weyl transformation of the form $e_z{}^m\delta e_m{}^z$ which has been omitted from Eq. (3.129) since it does not induce a motion in supermoduli space. Thus, in Wess-Zumino gauge, the super Beltrami differential may be decomposed as

$$\mu_K=\bar\theta(\mu_K^1+\theta\mu_K^0)\ ,$$

where

$$\mu_K^1=e_{\bar z}{}^m\frac{\partial e_m{}^z}{\partial m_K},\quad \mu_K^0=-\frac{\partial\chi_{\bar z}^+}{\partial m_K}\ . \qquad (3.130)$$

Clearly, μ_k^1 and μ_b^0 are even and correspond (for $\chi=0$) to the ordinary Beltrami differentials.[15]

From the duality of μ_K and Φ_J, it follows that their components are also naturally dual,

$$\langle\mu_K\mid\Phi_J\rangle=\langle\mu_K^0\mid\Phi_J^0\rangle+\langle\mu_K^1\mid\Phi_J^1\rangle\ , \qquad (3.131)$$

and for $\chi=0$ the ordinary Beltrami differentials μ_k^1 and μ_b^0 are dual to the holomorphic quadratic and $\frac{3}{2}$ differentials, respectively.

Finally, we introduce *super-quasiconformal vector fields* associated with *superquasiconformal transformations.* The superderivative of a super-quasiconformal vector field is to be identified with the super Beltrami differential, which lies in $T(s\mathcal{M}_h)$, and may be viewed as a deformation of the supercomplex structure,

$$(\mu_K)_-{}^z=\mathcal{D}_-V_K^z\ . \qquad (3.132)$$

It is again useful to restrict our attention to the case of Wess-Zumino gauge, and with Eq. (3.132) we find that V_K^z must be of the form

$$V_K^z=V_K^1+\theta V_K^0-\frac{i}{2}\theta\bar\theta AV_K^1$$

with

$$\mu_K^0=D_{\bar z}V_K^0+\tfrac{1}{2}\chi_{\bar z}{}^+D_zV_K^1+i\Lambda_-V_K^1\ ,$$
$$\mu_K^1=D_{\bar z}V_K^1+\tfrac{1}{2}\chi_{\bar z}{}^+V_K^0\ . \qquad (3.133)$$

For $\chi=0$, V_k^1 reduces to the ordinary quasiconformal vector field.

Super-quasiconformal transformations W can be defined as satisfying the *super Beltrami equation*

$$\mathcal{D}_-W=\mu\mathcal{D}_zW \qquad (3.134)$$

for a general super Beltrami differential $\mu=\sum\zeta_K\mu_K$. When $\chi=0$, it contains the ordinary Beltrami equation for the body component of W.

2. Superghost expression for superdeterminants

To represent the Faddeev-Popov superdeterminants, we introduce a ghost superfield C of U(1) weight -1 and an antighost superfield B of U(1) weight $\frac{3}{2}$, as well as their complex conjugates $\bar C$ and $\bar B$. We shall also assign ghost charge 1 to C and $\bar B$ and -1 to $\bar C$ and B. The relevant superghost action is

$$I_{\rm sgh}(C,B)=\frac{1}{2\pi}\int d^2\mathbf{z}\,E(B\mathcal{D}_-C+\bar B\mathcal{D}_+\bar C)\ . \qquad (3.135)$$

[15] μ_b^0 has also been termed a super Beltrami differential in the literature. We shall, however, reserve this name for μ_K.

Clearly, $I_{\rm sgh}$ is super-reparametrization, local U(1), and super Weyl invariant, provided B and C scale as $C=e^{\Sigma}\hat{C}$ and $B=e^{-(3/2)\Sigma}\hat{B}$. We introduce functional measures DC and DB through the metrics

$$\|\delta C\|^2=\int d^2{\bf z}\, E\delta\bar{C}\delta C \ ,\qquad \|\delta B\|^2=\int d^2{\bf z}\, E\delta\bar{B}\delta B \ , \tag{3.136}$$

each of which is invariant under ghost number rotations, super-reparametrizations, and U(1) transformations, but not under super Weyl rescalings. If we discard integrations over zero modes (denoted by primed fields), we have in a straightforward manner[16]

$$\int D(B'\bar{B}'C\bar{C})e^{-I_{\rm sgh}(C,B')}=({\rm sdet}\mathcal{P}_1^\dagger\mathcal{P}_1)^{1/2} \ . \tag{3.137}$$

This integral involving the first-order action $I_{\rm sgh}$ on the odd (C) and even (B) superfields may be understood by considering a toy example. Take the case of an odd ($C=c+\theta\gamma$) and an even ($B=\beta+\theta b$) supervariable. The ordinary integral is easily evaluated, and we find

$$\int dB \exp\left[i\int d\theta\, BC\right]=\int db\, d\beta\, e^{i(bc-\beta\gamma)} =2\pi i\delta(C) \ , \tag{3.138}$$

where $\delta(C)=\delta(c)\delta(\gamma)$. Thus, carrying out the B' and $\bar{B}'$ integrals in Eq. (3.137), one finds

$$\int D(B'\bar{B}')e^{-I_{\rm sgh}(C,B')}=\delta(\mathcal{D}_-C)\delta(\mathcal{D}_+\bar{C}) \ , \tag{3.139}$$

so that the C and $\bar{C}$ integrals produce precisely the Jacobian factor as given by Eq. (3.137).

To obtain a representation including the finite-dimensional determinants as well, we should integrate over the zero modes of B. This can be done by adding to the ghost action the coupling of B to super Beltrami differentials, since these are dual to the zero modes. To do so we introduce variables ζ_K (ζ_k's are odd, ζ_b's even) and evaluate the integral

$$\prod_K \int d^2\zeta_K \int D(B\bar{B}C\bar{C})\exp\left[-I_{\rm sgh}(C,B)+\sum_K \zeta_K\langle\mu_K\,|\,B\rangle+\bar{\zeta}_K\langle\bar{\mu}_K\,|\,\bar{B}\rangle\right] \tag{3.140}$$

in two different ways. First, by separating $B=B_0+B'$ into the zero-mode contribution B_0 and the non-zero-mode contribution B', we see that the term involving μ_K precisely couples only to B_0, whereas $I_{\rm sgh}$ depends only on B'. Thus the B_0 and B' integrals separate. The B' integral produces the infinite-dimensional superdeterminant as in Eq. (3.137). In the B_0 integral, we may decompose B_0 onto Φ_J (suitably normalized): $B_0=\sum_J \beta_J\Phi_J$, and since B_0 is even, β_a is odd and β_j even. The B_0 and ζ integrals then reduce to

$$\prod_{J,K}\int d\zeta_K d\beta_J \exp\left[\sum_{J,K}\zeta_K\langle\mu_K\,|\,\Phi_J\rangle\beta_J\right]=\frac{{\rm sdet}\langle\mu_K\,|\,\Phi_J\rangle}{{\rm sdet}\langle\Phi_K\,|\,\Phi_J\rangle^{1/2}} \tag{3.141}$$

after restoring the normalization. Our second way of evaluating Eq. (3.140) is to carry out the ζ integral first. Putting all together, we have

$$\left[\frac{{\rm sdet}\mathcal{P}_1^\dagger\mathcal{P}_1}{{\rm sdet}\langle\Phi_J\,|\,\Phi_K\rangle}\right]^{1/2}{\rm sdet}\langle\mu_K\,|\,\Phi_J\rangle=\int D(B\bar{B}C\bar{C})e^{-I_{\rm sgh}(C,B)}\prod_K|\,\delta(\langle\mu_K\,|\,B\rangle)\,|^2 \ . \tag{3.142a}$$

Since for $K=k$, $\langle\mu_K\,|\,B\rangle$ is odd, the δ function reduces to a linear function, so that equivalently

$$\prod_K\delta(\langle\mu_K\,|\,B\rangle)=\prod_k\langle\mu_k\,|\,B\rangle\prod_b\delta(\langle\mu_b\,|\,B\rangle) \ . \tag{3.142b}$$

Thus we arrive at a general formula for the scattering amplitudes in terms of the superghosts,[17]

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_h=\int_{s\mathcal{M}_h}d^2m_K\int D(XBC)\langle\langle V_1(k_1)\cdots V_n(k_n)\rangle\rangle\prod_b|\,\delta(\langle\mu_b\,|\,B\rangle)\,|^2\prod_k|\,\langle\mu_k\,|\,B\rangle\,|^2e^{-I} \ . \tag{3.143}$$

Here, I is the full action $I=I_m+I_{\rm sgh}$. As compared with the ghost formulation of the bosonic string, an unexpected novelty arises here. Whereas $\langle\mu_k\,|\,B\rangle$ amounts to an insertion of the operator B, the factors $\delta(\langle\mu_b\,|\,B\rangle)$ give rise to a new type of nonlocal insertion. We shall come back to this issue when dealing with the component formulation.

Further reformulation is possible when representing super Beltrami differentials in terms of superquasiconformal vector fields, through Eq. (3.132). Remarking that the B field is effectively holomorphic, we have

$$\langle\mu_K\,|\,B\rangle\sim\int d^2{\bf z}\,E\mathcal{D}_-(BV_K) \ .$$

Super-quasiconformal vector fields may be viewed as

[16]For the sake of definiteness, we shall only consider the case $h\geq 2$, where C has no zero modes. Otherwise, the C integration must be similarly restricted.

[17]Henceforth, we use the notation $d^2m_K=\prod_K d\bar{m}_K dm_K$.

"super-reparametrizations" with a discontinuity δV_K across a contour C_K. In that case, the inner products further reduce to (for $B=\beta+\theta b+\cdots$)

$$\langle \mu_K \mid B\rangle = \oint_{C_K} dz(\beta\delta V_K^0 + b\delta V_K^1) + \oint_{C_K} d\bar{z}\, \chi_{\bar{z}}{}^{+}\beta\delta V_K^1 \ . \tag{3.144}$$

We shall not make use of this formulation at present, and just point out that it should find use when dealing with the equivalence between the Polyakov first-quantized superstring, as we have discussed here, and Witten's string field-theoretic formulation of the superstring.

3. BRST symmetry

We begin by discussing the stress tensor. Supergeometry is specified by only six independent fields, and thus there are only six independent components of the stress tensor, defined through an infinitesimal change in the total action,

$$I = \frac{1}{2\pi}\int d^2\mathbf{z}\, E(\tfrac{1}{2}\mathcal{D}_- X^\mu \mathcal{D}_+ X_\mu + B\mathcal{D}_- C + \bar{B}\mathcal{D}_+\bar{C}) \ , \tag{3.145}$$

$$\delta I \equiv \frac{1}{2\pi}\int d^2\mathbf{z}\, E(H_+{}^{+}T_+{}^{+} + H_+{}^{z}T_z{}^{+} + H_-{}^{z}T_z{}^{-} + \text{c.c.}) \ . \tag{3.146}$$

The full action I is U(1) and super Weyl invariant, so we must have $T_+{}^{+}=0$, and since it is invariant under super-reparametrizations $\delta V^{\pm}$, we also have $T_z{}^{+}=0$ at the classical level.[18] These symmetries will also be implemented at the quantum level (in the critical dimension), so we shall completely ignore the components $T_+{}^{+}$ and $T_z{}^{+}$ and set them to zero. We shall also denote $T_z{}^{-}=T$ and call this the stress tensor. Invariance under super-reparametrizations δV^z implies that T is conserved,

$$\mathcal{D}_-^{3/2}T = 0 \ . \tag{3.147}$$

It is sometimes convenient to consider the matter (T_m) and superghost (T_{sgh}) contributions separately; they are given by

$$\begin{aligned} T &= T_m + T_{\text{sgh}} \ , \\ T_m &= -\tfrac{1}{2}\mathcal{D}_+ X^\mu \mathcal{D}_+^2 X_\mu \ , \\ T_{\text{sgh}} &= -C\mathcal{D}_+^2 B + \tfrac{1}{2}\mathcal{D}_+ C\mathcal{D}_+ B - \tfrac{3}{2}(\mathcal{D}_+^2 C)B \ , \end{aligned} \tag{3.148}$$

and are classically conserved.

Once the local gauge symmetries have been fixed and Faddeev-Popov ghosts introduced, the presence of the original symmetries is revealed by the existence of BRST symmetry. The total action I is indeed invariant under

$$\begin{aligned} \delta X^\mu &= \lambda C\mathcal{D}_+^2 X^\mu - \tfrac{1}{2}\lambda\mathcal{D}_+ C\mathcal{D}_+ X^\mu + \text{c.c.} \ , \\ \delta C &= \lambda C\mathcal{D}_+^2 C - \tfrac{1}{4}\lambda\mathcal{D}_+ C\mathcal{D}_+ C \ , \\ \delta B &= -\lambda T \ , \end{aligned} \tag{3.149}$$

where λ is an odd constant parameter. Associated with this symmetry is a current of weight $\frac{1}{2}$,

$$j_{\text{BRST}} = C(T_m + \tfrac{1}{2}T_{\text{sgh}}) - \tfrac{3}{4}\mathcal{D}_+(C(\mathcal{D}_+ C)B) \ , \tag{3.150}$$

which is conserved: $\mathcal{D}_- j_{\text{BRST}}=0$. It was pointed out by Friedan, Martinec, and Shenker (1986) that the superghost system by itself also possesses an additional U(1) symmetry, making it into an $N=2$ superconformal algebra. The associated U(1) current

$$j_{\text{U(1)}} = 2(\mathcal{D}_+ B)C + 3B\mathcal{D}_+ C \tag{3.151}$$

is conserved: $\mathcal{D}_{\bar{z}} j_{\text{U(1)}}=0$.

The presence of BRST symmetry implies the existence of certain Ward identities for the correlation functions, assuming that these are taken with respect to a (physical) BRST-invariant vacuum. In the case of the bosonic string we presented two somewhat distinct methods for handling these Ward identities. In the first one, the BRST charge was written as a line integral and analyticity properties of the correlation functions were used to "pull off the contour" and rewrite the full contribution as a total derivative over moduli space. The second method did not rely on such analyticity properties and has a wider range of applicability, though in the case where the correlation functions possess analyticity properties, these are not readily translated into this langauge.

For the superstring, as we shall see explicitly in Sec. VII, the superghost correlation functions are meromorphic, but they possess in addition to the expected poles some spurious poles, which in general have to be taken into account in the analyticity arguments before correct conclusions can be drawn. Maybe the use of supercontour integrals and superanalyticity on the superworldsheet can get around this problem. For the time being, we shall formulate the BRST Ward identities using the more general functional treatment, where no analyticity properties are assumed. One derives such identities on the generating functional, and by differentiating with respect to the sources, one can obtain them for any correlation function.

The starting point is the generating functional

$$Z(X^*,B^*,C^*) = \int D(XBC)\exp[-I(X,B,C) + I_s(X,B,C;X^*,B^*,C^*)] \ , \tag{3.152}$$

where I is the total action of Eq. (3.145) and I_s couples the external sources X^*, B^*, and C^* to the fields X, B, and C in a super-reparametrization and local-U(1)-invariant way:

$$I_s = \int d^2\mathbf{z}\, E(X^*X + B^*B + C^*C + \bar{B}^*\bar{B} + \bar{C}^*\bar{C}) \ . \tag{3.153}$$

Correlation functions are obtained by taking successive functional derivatives of Z. We introduce the notation[19]

[18]Compare with the bosonic string where Weyl invariance implies that $T_{z\bar{z}}=0$.

[19]Henceforth, we suppress the μ index on the X field.

$$\hat{X}=\frac{1}{E}\frac{\delta}{\delta X^*},\quad \hat{B}=\frac{1}{E}\frac{\delta}{\delta B^*},\quad \hat{C}=\frac{1}{E}\frac{\delta}{\delta C^*}. \tag{3.154}$$

We shall also need the functional derivative with respect to the supergeometry changes $H_-{}^z$,

$$\hat{H}=\frac{1}{E}\frac{\delta}{\delta H_-{}^z}. \tag{3.155}$$

For example, the partition function, according to Eq. (3.143), becomes[20]

$$Z_h=\int_{s\mathcal{M}_h} d^2m_K \prod_K |\delta(\langle\mu_K|\hat{B}\rangle)|^2 Z(X^*,B^*,C^*)\Big|_{*=0}, \tag{3.156}$$

so that the operators with hats effectively play the role of the quantum operators associated with the fields.

The BRST Ward identities are derived on the assumption that the measure $D(XBC)$ is invariant under BRST transformations (3.149), which will be true at the quantum level only in the critical dimension. We then define the BRST operator

$$\lambda\hat{Q}_{\mathrm{BRST}}Z(X^*,B^*,C^*) = \int_{s\mathcal{M}_h} d^2m_K \int D(XBC)(\delta_{\mathrm{BRST}}I_s)e^{-I+I_s}. \tag{3.157}$$

A little algebra gives

$$\hat{Q}_{\mathrm{BRST}}=\int d^2\mathbf{z}\, E[X^*(\hat{C}\mathcal{D}_+^2\hat{X}-\tfrac{1}{2}\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{X})-B^*\hat{H} - C^*(\hat{C}\mathcal{D}_+^2\hat{C}-\tfrac{1}{4}\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{C})+\text{c.c.}], \tag{3.158}$$

which yields almost the same BRST transformation laws for the operators $\hat{X}$, $\hat{B}$, and $\hat{C}$ as given in Eq. (3.149):

$$\begin{aligned}\delta\hat{X}&=[\hat{X},\lambda\hat{Q}_{\mathrm{BRST}}]\\ &=\lambda\hat{C}\mathcal{D}_+^2\hat{X}-\tfrac{1}{2}\lambda\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{X}+\text{c.c.},\\ \delta\hat{C}&=[\hat{C},\lambda\hat{Q}_{\mathrm{BRST}}]=\lambda\hat{C}\mathcal{D}_+^2\hat{C}-\tfrac{1}{4}\lambda\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{C},\\ \delta\hat{B}&=[\hat{B},\lambda\hat{Q}_{\mathrm{BRST}}]=-\lambda\hat{H}.\end{aligned} \tag{3.159}$$

As an interesting application, we may evaluate the BRST behavior of an insertion occurring in the expressions for the amplitudes[21]

$$[\langle\mu_K|\hat{B}\rangle,\lambda\hat{Q}_{\mathrm{BRST}}]=\lambda\left\{(-)^K\frac{\partial}{\partial m_K}+\int d^2\mathbf{z}\,E_z\int d^2\mathbf{w}\,E_w B^*(z)[\hat{H}(z)\mu_K(w)]\hat{B}(w)\right\}, \tag{3.160}$$

where we have used the fact that

$$\langle\mu_K|\hat{H}\rangle=\frac{\partial}{\partial m_K}. \tag{3.161}$$

The (anti) commutator of this object with another insertion vanishes in view of the integrability conditions (3.128),

$$[\langle\mu_L|\hat{B}\rangle,[\langle\mu_K|\hat{B}\rangle,\lambda\hat{Q}_{\mathrm{BRST}}]]=\lambda(-)^{L+1}\left\langle\frac{\partial\mu_K}{\partial m_L}+(-)^{KL}\frac{\partial\mu_L}{\partial m_K}\middle|\hat{B}\right\rangle=0. \tag{3.162}$$

We also have

$$[\delta(\langle\mu_K|\hat{B}\rangle),\lambda\hat{Q}_{\mathrm{BRST}}]=[\langle\mu_K|\hat{B}\rangle,\lambda\hat{Q}_{\mathrm{BRST}}]\delta'(\langle\mu_K|\hat{B}\rangle), \tag{3.163}$$

where the ordering of δ' and [ ] on the right-hand side is immaterial in view of Eq. (3.162). With the help of Eq. (3.162) once more, we can now permute the BRST operator through all insertions,

$$\left[\prod_K\delta(\langle\mu_K|\hat{B}\rangle),\lambda\hat{Q}_{\mathrm{BRST}}\right]=\sum_{K'=1}^{5h-5}[\langle\mu_{K'}|\hat{B}\rangle,\lambda\hat{Q}_{\mathrm{BRST}}]\prod_{K=1}^{K'-1}\delta(\langle\mu_K|\hat{B}\rangle)\delta'(\langle\mu_{K'}|\hat{B}\rangle)\prod_{K=K'+1}^{5h-5}\delta(\langle\mu_K|\hat{B}\rangle). \tag{3.164}$$

To deal with scattering amplitudes, we have to insert vertex operators for physical states. Furthermore, we must show that a total BRST change in any vertex operator—which simply amounts to a gauge transformation in field theory langauge—produces a vanishing contribution.

The physical vertex operators for the emission or absorption of bosonic particles in the functional formulation can be taken to be super-reparametrization-, local-U(1)-, and super-Weyl-invariant vertex operators of the Polyakov string (to be discussed fully in Sec. VIII) without any B or C insertions, and they are thus of the form $V_i(\hat{X},H)$. It is not hard to see that they are automatically BRST invariant (in the critical dimension) in the following sense:

$$[\lambda\hat{Q}_{\mathrm{BRST}},V_i(\hat{X},H)]Z(X^*,B^*,C^*)=0. \tag{3.165}$$

To show this we need the Ward identities of the generating functional under super-reparametrizations, local U(1), and

[20]The subscript $*=0$ sets all sources to zero.

[21]$(-)^K=1$ when $K=k$ and -1 when $K=b$.

super Weyl symmetry. All of these are nonanomalous in the critical dimension, as long as the sources remain orthogonal to the zero modes of their corresponding fields. For example, it is straightforward to derive the super-reparametrization Ward identity

$$\int d^2z\, E(\mathcal{D}_- V^z[\hat{H},V_i]-\tfrac{1}{2}\mathcal{D}_+ V^z V_i'\hat{X}-V^z\mathcal{D}_+^2 V_i'\hat{X}+\tfrac{1}{2}V^+\mathcal{D}_+ V_i'\hat{X})Z(X^*,B^*,C^*)=0 \tag{3.166}$$

valid for arbitrary fields V^z and V^+. For later convenience, we have added the effect of a supplementary super Weyl and local U(1) transformation of the second term. Evaluating Eq. (3.165) explicitly, we get

$$\int d^2z\, E[-B^*[\hat{H},V_i]-(\hat{C}\mathcal{D}_+^2\hat{X}-\tfrac{1}{2}\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{X})V_i'(\hat{X},H)]Z(X^*,B^*,C^*)=0\ . \tag{3.167}$$

Finally, we use the Schwinger-Dyson equation,

$$(B^*-\mathcal{D}_-\hat{C})Z(X^*,B^*,C^*)=0\ ,$$

in order to replace B^* in the first term of Eq. (3.167) by $\mathcal{D}_-\hat{C}$. Furthermore, since V^z and V^+ were arbitrary in Eq. (3.166), we may choose $V^z=\lambda\hat{C}$ and $V^+=\lambda\mathcal{D}_+\hat{C}$ and add Eq. (3.166) to (3.167). The exact cancellation shows that Eq. (3.165) holds, so that any super-reparametrization-, local U(1)-, and super-Weyl-invariant vertex V_i is also BRST invariant.

To show decoupling of BRST charges, let us consider the amplitude with $n-1$ physical (BRST-invariant) vertex operators $V_1,\ldots,V_{n-1}$ and one insertion of the BRST transform of an arbitrary operator V_n,

$$\langle V_1\cdots V_{n-1}[\lambda Q_{\rm BRST},V_n]\rangle_h=\int_{s\mathcal{M}_h} d^2m_K\prod_K |\delta(\langle\mu_K|\hat{B}\rangle)|^2\hat{V}_1\cdots\hat{V}_{n-1}[\lambda\hat{Q}_{\rm BRST},\hat{V}_n]Z(X^*,B^*,C^*)\Big|_{*=0}\ . \tag{3.168}$$

The BRST invariance of $\hat{V}_i$, $i=1,\ldots,n-1$ and of the generating functional allows us to move $\lambda\hat{Q}_{\rm BRST}$ just to the right of all δ-function insertions. With the help of Eq. (3.164), we can bring the resulting commutator $[\langle\mu_{K'}|\hat{B}\rangle,\lambda\hat{Q}_{\rm BRST}]$ completely to the left. But now the sources should be set to zero, and only the derivative with respect to m_K remains from Eq. (3.160), so that

$$\langle V_1\cdots V_{n-1}[Q_{\rm BRST},V_n]\rangle_h=\int_{s\mathcal{M}_h} d^2m_K\sum_{K'=1}^{5h-5}\frac{\partial}{\partial m_{K'}}W_{K'} \tag{3.169}$$

with

$$W_{K'}=\prod_{K=1}^{K'-1}\delta(\langle\mu_K|\hat{B}\rangle)\delta'(\langle\mu_{K'}|\hat{B}\rangle)\prod_{K=K'+1}^{5h-5}\delta(\langle\mu_K|\hat{B}\rangle)\prod_{K=1}^{5h-5}\overline{\delta(\langle\mu_K|\hat{B}\rangle)}\hat{V}_1\cdots\hat{V}_n Z(X^*,B^*,C^*)\Big|_{*=0}$$

Thus the insertion of BRST changes in arbitrary operators produces total derivatives on supermoduli space. The total contributions then arise only from evaluating $W_{K'}$ at the boundary of moduli space. If the string theory satisfies all its equations of motion, i.e., the background space-time is a solution to the "string field equations of motion," then such contributions may be expected to vanish. However, when this is not the case cancellation may be required with effects on surfaces of different topology.

The use of superfield superghosts was proposed by Friedan, Martinec, and Shenker (1986) and further developed by Martinec (1987).

K. Chiral splitting in the component formalism

Though the expressions for the amplitudes obtained in the previous section are complete, one may wish to render them yet more explicit by working in the component formalism. Actually this is where the calculation for these amplitudes was performed in the first place. In this section we shall treat the case of the type-II superstring, postponing the discussion of the heterotic string to Sec. III.N.

Upon choosing Wess-Zumino gauge, we find that the superspace action (3.40) reduces to Eq. (3.1). We shall recall it here for convenience and display its dependence on complex (chiral) fields explicitly. We shall also drop the term proportional to the Euler characteristic, as well as the one involving the auxiliary field F,

$$I_m=I_x+I_\psi+I_m^1+I_m^2\ ,$$

where

$$\begin{aligned}
I_x&=\frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,D_z x^\mu D_{\bar z}x^\mu\ ,\\
I_\psi&=\frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(-\psi_+^\mu D_{\bar z}\psi_+^\mu-\psi_-^\mu D_z\psi_-^\mu)\ ,\\
I_m^1&=\frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(\chi_{\bar z}{}^+\psi_+^\mu D_z x^\mu+\chi_z{}^-\psi_-^\mu D_{\bar z}x^\mu)\ ,\\
I_m^2&=\frac{1}{8\pi}\int_M d^2\xi\sqrt{g}\,\chi_z{}^-\chi_{\bar z}{}^+\psi_+^\mu\psi_-^\mu\ .
\end{aligned} \tag{3.170}$$

This matter action could now be considered as a supergravity theory in its own right. For string theory, quantization would require integrating over the x^μ, ψ^μ, g_{mn}, and χ_m fields in a reparametrization-invariant, local supersymmetric, and Weyl-invariant way. The difficulty is that it is impossible to define a workable measure for the component fields that is local and supersymmetric; indeed, the framework in which local supersymmetry is manifest is precisely superspace. Thus, instead of taking the action (3.170) as our starting point and quantizing it directly, we shall rather begin with the superspace formulation of the previous sections and project it down onto Wess-Zumino gauge. Since it is most convenient to perform such gauge choices in a local quantum field theory, we see that the superghost formalism is most practical in this respect. Notice that the choice of the Wess-Zumino gauge can always be implemented in a purely algebraic way. After this has been done, the symmetries are those described in Sec. III.C.

We now restrict the superghost action $I_{\rm sgh}(C,B)$ to Wess-Zumino gauge as well. To this end, we decompose the ghost superfields into components,

$$
\begin{aligned}
B&=\beta+\theta b+\bar\theta B_2+i\theta\bar\theta B_3 \ ,\\
C&=c+\theta\gamma+\bar\theta C_2+i\theta\bar\theta C_3 \ .
\end{aligned}
\tag{3.171}
$$

We also restrict χ_m to be γ-traceless, as may be done in the critical dimension where the super Weyl anomaly cancels. One then finds

$$
I_{\rm sgh}=\frac{1}{2\pi}\int_M d^2\xi\sqrt{g}\left[-iB_3C_2+B_2\left[iC_3+\frac{i}{2}Ac\right]+b(D_{\bar z}c+\tfrac{1}{2}\chi_{\bar z}{}^{+}\gamma)+\beta\left[\frac{3i}{4}AC_2+\tfrac{1}{2}\chi_{\bar z}{}^{+}D_zc+D_{\bar z}\gamma+i\Lambda_{-}c\right]\right]+\text{c.c.}
\tag{3.172}
$$

It remains to evaluate the contribution from the δ functions on $\langle\mu_K\mid B\rangle$ in Eq. (3.142) to have the full ghost expression. Since in Wess-Zumino gauge μ_k is given by Eq. (3.129), we see that B_2 and B_3 (and their complex conjugates) never contribute to these inner products, and we have

$$
\langle\mu_K\mid B\rangle=\langle\mu_K^1\mid b\rangle+\langle\mu_K^0\mid\beta\rangle \ .
\tag{3.173}
$$

Thus in the full B-C integrals in Eq. (3.142), the fields B_2, B_3, C_2, C_3 and their complex conjugates are auxiliary and never carry any derivatives. They may be integrated out explicitly, and ultralocality here says that the only effect will be a super area term, whose coefficient is determined by super Weyl symmetry and is thus immaterial.

We end up with the following expression for the super Faddeev-Popov determinants in Wess-Zumino gauge:

$$
\left[\frac{\text{sdet}\mathcal{P}_1^\dagger\mathcal{P}_1}{\text{sdet}\langle\Phi_J\mid\Phi_K\rangle}\right]^{1/2}\text{sdet}\langle\mu_K\mid\Phi_J\rangle=\int D(bc\beta\gamma)e^{-I_{\rm sgh}}\prod_k|\langle\mu_k\mid B\rangle|^2\prod_b|\delta(\langle\mu_b\mid B\rangle)|^2 \ ,
\tag{3.174}
$$

where it is now understood that the field B in the products is restricted to $B=\beta+\theta b$, the superghost action takes on the simplified form

$$
I_{\rm sgh}=I^0_{\rm sgh}+I^1_{\rm sgh} \ ,
$$

where

$$
\begin{aligned}
I^0_{\rm sgh}&=\frac{1}{2\pi}\int_M d^2\xi\sqrt{g}\,(bD_{\bar z}c+\beta D_{\bar z}\gamma+\text{c.c.}) \ ,\\
I^1_{\rm sgn}&=-\frac{1}{2\pi}\int_M d^2\xi\sqrt{g}\,(\chi_{\bar z}{}^{+}S_{\rm gh}+\chi_z{}^{-}\bar S_{\rm gh}) \ ,
\end{aligned}
\tag{3.175}
$$

and the ghost supercurrent that is the θ-independent piece of the super stress tensor $T_{\rm sgh}$ of Eq. (3.148) is given by

$$
S_{\rm gh}=\tfrac{1}{2}b\gamma-\tfrac{3}{2}\beta D_zc-(D_z\beta)c \ .
\tag{3.176}
$$

Actually, we shall sometimes make use of the full current

$$
S=-\tfrac{1}{2}\psi_+^\mu D_zx^\mu+S_{\rm gh} \ ,
\tag{3.177}
$$

which is only the χ-independent part of the full supercurrent (the θ-independent component of the stress tensor T). We shall see later on that it is, however, all we need.

We are now in a position to express the general superstring amplitude to h-loop order ($h\geq 2$) as an integral over supermoduli, formulated in components

$$
\langle V_1(k_1)\cdots V_n(k_n)\rangle_h=\int_{s\mathcal{M}_h}d^2m_K\int D(x\psi bc\beta\gamma)\prod_k|\langle\mu_k\mid B\rangle|^2\prod_b|\delta(\langle\mu_b\mid B\rangle)|^2V_1(k_1)\cdots V_n(k_n)e^{-I} \ ,
\tag{3.178}
$$

where $I=I_m+I_{\rm sgh}$ is the total action in components and $B=\beta+\theta b$.

1. Chiral splitting of the matter integrals

In Sec. II.B, we have seen that physical vertex operators (for bosonic particles) do not depend on ghosts or superghosts. Hence the amplitude (3.178) exactly as in Eq. (3.174) is "chirally split" in terms of the chiral ghost fields $bc\beta\gamma$ and $\bar b\bar c\bar\beta\bar\gamma$ in the sense that there is no coupling between the opposite chiralities of these fields. This will be a crucial property in defining both the heterotic and type-II strings and will manifest itself under the form of superholomorphic factorization, as we shall see in Sec.

VII. However, it is clear that this chiral factorization does not manifestly hold for the matter part. Of course, this could not have been expected in the first place, since the x field is real and not chiral. Furthermore, the term I_m^2, bilinear in χ, couples ψ_+ to ψ_- and seems to spoil chirality. One of the main tasks of this section will be to formulate a modified version of chiral splitting which holds for the full amplitude.

To display chiral splitting of the matter part of the functional integrals, one must integrate out the x field in any amplitude. For simplicity we shall not consider full vertex operators, but just insert the universal factors $e^{ik\cdot X}$ required by translation invariance, located at different points on the surface. This may be thought of as a tachyon operator whose position is not yet integrated over. It is only technically harder to deal with the insertion of full vertex operators. In Wess-Zumino gauge where auxiliary fields have been integrated out, we have

$$e^{ik\cdot X(z)}=e^{ik\cdot x(z)}e^{ik\cdot\theta\psi_+(z)}e^{ik\cdot\bar\theta\psi_-(z)} . \tag{3.179}$$

It is clear that the dependence on ψ_+ and ψ_- is already chirally split, so we shall deal with it later on. Notice that the second and third exponentials on the right-hand side are complex conjugates of one another only when k^μ is purely imaginary. Of course, physically k^μ is rather a real vector, but we shall also see later on that from several points of view k^μ should be analytically continued to imaginary values.

Thus we are ultimately interested in the integral

$$\mathcal{A}_x=\int Dx\prod_{i=1}^n e^{ik_i^\mu x^\mu(z_i)}e^{-I_x-I_m^1-I_m^2} , \tag{3.180}$$

leaving the Dirac fermion $\psi_\pm$ integrals for later. We have, however, included the I_m^2 term here, because it will naturally cancel some of the x integrals. The Green's function for the x field is

$$G(z,w)=\langle x(z)x(w)\rangle , \tag{3.181}$$

which is, however, not Weyl invariant as explained in Sec. II.G, and it is appropriate to define the Weyl-invariant combination $F(z,w)$,

$$\begin{aligned}-\ln F(z,w)= G(z,w)+\tfrac{1}{2}\ln\rho(z)+\tfrac{1}{2}\ln\rho(w)\\ -\tfrac{1}{2}G_R(z,z)-\tfrac{1}{2}G_R(w,w) .\end{aligned} \tag{3.182}$$

Furthermore, recall that $F(z,w)$ has a very simple decomposition,

$$\begin{aligned}\ln F(z,w)= \ln|E(z,w)|^2\\ -2\pi\,\mathrm{Im}\int_z^w\omega_I(\mathrm{Im}\Omega)_{IJ}^{-1}\mathrm{Im}\int_z^w\omega_J ,\end{aligned} \tag{3.183}$$

where $E(z,w)$ is the prime form and Ω_{IJ} the period matrix.

The Gaussian integral is now easily performed, and one gets

$$\mathcal{A}_x=(2\pi)^{10}\delta(k)\left[\frac{8\pi^2\det'\Delta}{\int_M d^2\xi\sqrt{g}}\right]^{-5}e^{\mathcal{K}^0+\mathcal{K}+\mathcal{K}'} , \tag{3.184}$$

where

$$\begin{aligned}\mathcal{K}^0&=-\tfrac{1}{2}\sum_{ij}k_i^\mu k_j^\mu\langle x(z_i)x(z_j)\rangle ,\\ \mathcal{K}&=-i\sum_i k_i^\mu\langle x^\mu(z_i)I_m^1\rangle ,\\ \mathcal{K}'&=\tfrac{1}{2}\langle I_m^1 I_m^1\rangle-I_m^2 .\end{aligned} \tag{3.185}$$

The next step is to single out the ingredients that are not manifestly split. They will be expressed in terms of correlation functions of the field σ_I^μ where[22]

$$\sigma_I^\mu=\frac{1}{4\pi}\int d^2z\,\chi_{\bar z}{}^{+}(z)\psi_+^\mu(z)\omega_I(z) . \tag{3.186}$$

Thus we find, using Eq. (3.182) and momentum conservation,

$$\begin{aligned}\mathcal{K}^0&=\mathcal{L}_+^0+\mathcal{L}_-^0+2\pi\sum_{ij}k_i^\mu k_j^\mu\mathrm{Im}\int_P^{z_i}\omega_I(\mathrm{Im}\Omega)_{IJ}^{-1}\mathrm{Im}\int_P^{z_j}\omega_J ,\\ \mathcal{K}&=\mathcal{L}_++\mathcal{L}_--4\pi i\,\mathrm{Im}\sigma_I^\mu(\mathrm{Im}\Omega)_{IJ}^{-1}\sum_i k_i^\mu\mathrm{Im}\int_P^{z_i}\omega_J ,\\ \mathcal{K}'&=\mathcal{L}'_++\mathcal{L}'_--2\pi\,\mathrm{Im}\sigma_I^\mu(\mathrm{Im}\Omega)_{IJ}^{-1}\mathrm{Im}\sigma_J^\mu ,\end{aligned} \tag{3.187}$$

where P is an arbitrary point on the worldsheet. The combinations $\mathcal{L}_+^0$, $\mathcal{L}_+$, and $\mathcal{L}'_+$ depend analytically on the z_i and on Ω_{IJ} and involve only the chiral fields ψ_+^μ,

$$\mathcal{L}_+^0=\sum_{i<j}k_i^\mu k_j^\mu\ln E(z_i,z_j) \tag{3.188}$$

and

$$\begin{aligned}\mathcal{L}_+&=\frac{i}{4\pi}\sum_{i=1}^n k_i^\mu\int d^2z\,\chi_{\bar z}{}^{+}(z)\psi_+^\mu(z)\partial_z\ln E(z,z_i) ,\\ \mathcal{L}'_+&=-\frac{1}{32\pi^2}\int d^2z\int d^2w\,\chi_{\bar z}{}^{+}\psi_+(z)\chi_{\bar w}{}^{+}\psi_+(w)\\ &\quad\times\partial_z\partial_w\ln E(z,w) .\end{aligned} \tag{3.189}$$

In practice, the expression $\exp(\mathcal{L}_+^0+\mathcal{L}_++\mathcal{L}'_+)$ can be viewed as resulting from contractions of an effectively chiral field $x_+(z)$ with effective propagator

$$\langle x_+(z)x_+(w)\rangle=-\ln E(z,w) , \tag{3.190}$$

so that

$$\exp(\mathcal{L}_+^0+\mathcal{L}_++\mathcal{L}'_+)=\langle e^{-I_m^1+ik_i^\mu x_+^\mu(z_i)}\rangle . \tag{3.191}$$

Finally, $\mathcal{L}_-^0$, $\mathcal{L}_-$, and $\mathcal{L}'_-$ are the complex conjugates of $\mathcal{L}_+^0$, $\mathcal{L}_+$, and $\mathcal{L}'_+$ with the understanding that k_i^μ is taken to be purely imaginary.

Returning to Eq. (3.184), the amplitude $\mathcal{A}_x$ can be rewritten as

[22] In the remainder of this section, the lower index on $\chi_{\bar z}{}^{+}$ is now an Einstein index, and repeated $I,J,\ldots$ indices are summed over.

$$\mathcal{A}_x = (2\pi)^{10}\delta(k)\left[\frac{8\pi^2\det'\Delta}{\int_M d^2\xi\sqrt{g}\,\det \mathrm{Im}\Omega}\right]^{-5} \times\exp(\mathcal{L}_+^0+\mathcal{L}_-^0+\mathcal{L}_++\mathcal{L}'_++\mathcal{L}_-+\mathcal{L}'_-)\mathcal{A}'_x\ , \tag{3.192}$$

and the remaining amplitude $\mathcal{A}'_x$ is gotten by collecting the pieces that are not yet manifestly chirally split. For later convenience we have rearranged a factor of $\det \mathrm{Im}\Omega$. Thus $\mathcal{A}'_x$ is given by

$$\mathcal{A}'_x=(\det \mathrm{Im}\Omega)^{-5}\exp\left[-2\pi\left[\mathrm{Im}\sigma_I^\mu+i\sum_{i=1}^n k_i^\mu \mathrm{Im}\int_P^{z_i}\omega_I\right](\mathrm{Im}\Omega)^{-1}_{IJ}\left[\mathrm{Im}\sigma_J^\mu+i\sum_{j=1}^n k_j^\mu \mathrm{Im}\int_P^{z_j}\omega_J\right]\right]\ . \tag{3.193}$$

We previously indicated a good reason for taking the external momenta purely imaginary. We now see that if k_i^μ are all imaginary, $\mathcal{A}'_x$ admits a remarkable representation generalizing the one encountered in Sec. II.G:

$$\mathcal{A}'_x=\int_{\Im} dp_I^\mu\left|\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi p_I^\mu\left[\sigma_I^\mu+i\sum_{i=1}^n k_i^\mu\int_P^{z_i}\omega_I\right]\right]\right|^2\ . \tag{3.194}$$

Here p_I^μ represent the internal loop momenta, and for consistency they have been analytically continued to imaginary values as well—this has been indicated by the subscript $\Im$ to the integral. Of course, the integral would not be convergent, so it should be symbolically understood: the absolute value square is taken with p_I^μ imaginary, but to evaluate the integral one must analytically continue to real p_I^μ.

The combination involving $\det'\Delta$ admits a splitting in terms of left- and right-movers on the Riemann surface as well (up to an anomaly that will ultimately be cancelled, as explained in Secs. VII.A and VII.D),

$$\frac{8\pi^2\det'\Delta}{\int_M d^2\xi\sqrt{g}\,\det \mathrm{Im}\Omega}=|Z_\Delta(\Omega)|^4\ . \tag{3.195}$$

Taking this into account, it becomes transparent that the full amplitude—for fixed internal, imaginary momenta p_I^μ—has been split (or factorized) as a product of an expression involving chiral operators ψ_+ and holomorphic $\mathbf{z}_i=(z_i,\theta_i)$ times its complex conjugate.

$$\mathcal{A}_x=(2\pi)^{10}\delta(k)\int_{\Im} dp_I^\mu \mathcal{F}_\nu(\mathbf{z}_i,\psi_+,\Omega,\chi;p_I^\mu)\overline{\mathcal{F}_\nu(\mathbf{z}_i,\psi_-,\Omega,\chi;p_I^\mu)}\ , \tag{3.196}$$

where the operator $\mathcal{F}_\nu$ only depends on ψ_+, $\mathbf{z}_i$, and Ω and not on ψ_-, $\bar{\mathbf{z}}_i$, or $\bar\Omega$,

$$\mathcal{F}_\nu(\mathbf{z}_i,\psi_+,\Omega,\chi;p_I^\mu)=[Z_\Delta(\Omega)]^{-10}\prod_{i<j}E(z_i,z_j)^{k_i\cdot k_j}e^{\mathcal{L}_++\mathcal{L}'_+}\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi p_I^\mu\left[\sigma_I^\mu+i\sum_i k_i^\mu\int_P^{z_i}\omega_I\right]\right]\ . \tag{3.197}$$

In formulating the type-II superstring, it was necessary to sum separately over the spin structures of left and right chiralities. This can now be easily achieved by evaluating the expectation value for the ψ_+ and ψ_- fields separately on each chiral component, each with its own spin structure. The two halves may then be brought back together for the same value of p_I^μ and the p_I^μ integral carried out. Thus the amplitude for different left- and right-spin structures ν and $\bar\nu$ is a simple generalization of Eq. (3.196),

$$\mathcal{A}_x=(2\pi)^{10}\delta(k)\int_{\Im} dp_I^\mu \mathcal{F}_\nu(\mathbf{z}_i,\psi_+,\Omega,\chi;p_I^\mu)\overline{\mathcal{F}_{\bar\nu}(\mathbf{z}_i,\psi_-,\Omega,\chi;p_I^\mu)}\ . \tag{3.198}$$

This entirely defines the matter contribution to the type-II superstring amplitudes involving only exponential insertions. The contributions of higher vertex operator insertions (containing in addition derivatives of x) can be similarly evaluated, and one arrives at an expression like (3.196), with $\mathcal{F}$ still chiral, but now also dependent on the derivative insertions. We shall work out the amplitudes for the scattering of massless particles for tree level in Sec. III.L and one-loop level in Sec. III.M.

Next, we must evaluate the amplitude for the full matter contribution, gotten by integrating out the Dirac fermion fields ψ_+ and ψ_-,

$$\mathcal{A}_m=\int D\psi_+D\psi_-\mathcal{A}_x e^{-I_\psi}\prod_{i=1}^n \exp[ik_i^\mu\theta_i\psi_+^\mu(z_i)+ik_i^\mu\bar\theta_i\psi_-^\mu(z_i)]\ , \tag{3.199}$$

where it is understood that ψ_+^μ and ψ_-^μ are endowed with spin structures ν and $\bar\nu$, respectively. With the help of Eq. (3.198) we may rewrite this expression,

$$\mathcal{A}_m=(2\pi)^{10}\delta(k)\int_{\Im} dp_I^\mu \mathcal{C}_\nu(\mathbf{z}_i,\Omega,\chi;p_I^\mu)\overline{\mathcal{C}_{\bar\nu}(\mathbf{z}_i,\Omega,\chi;p_I^\mu)}\ , \tag{3.200}$$

where

$$\mathcal{C}_\nu(\mathbf{z}_i,\Omega,\chi;p_I^\mu)=\mathcal{C}'_\nu[Z_\Delta(\Omega)]^{-10}\prod_{i<j}E(z_i,z_j)^{k_i\cdot k_j}\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi i p_I^\mu\sum_{i=1}^n k_i^\mu\int_P^{z_i}\omega_I\right],$$
$$\mathcal{C}'_\nu=\int D\psi_+^\mu e^{-I_{\psi_+}+\mathcal{L}_++\mathcal{L}'_+}\prod_{i=1}^n e^{ik_i^\mu\theta_i\psi_+^\mu(z_i)}e^{2\pi p_I^\mu\sigma_I^\mu}\ . \tag{3.201}$$

When the spin structure is even, there are generically no zero modes to the Dirac operator, and the Dirac propagator is given by the Szegö kernel (Secs. VI.F and VII.C),

$$S_\nu(z,w)=-\langle\psi_+(z)\psi_+(w)\rangle_\nu=\frac{\vartheta[\nu]\left[\int_w^z\omega_I,\Omega\right]}{E(z,w)\vartheta[\nu](0,\Omega)}\ , \tag{3.202}$$

which is meromorphic in z and w, and analytic in Ω. As a consequence, the reduced amplitudes $\mathcal{C}_\nu$ of Eq. (3.201) are analytic functions of $\mathbf{z}_i,\Omega_{IJ}$ and they depend only on $\chi_{\bar z}{}^+$.

When the spin structure is odd, there is generically one zero mode $h_\nu(z)$ to the Dirac operator, and the Dirac propagator is not uniquely defined. One choice is to take the propagator orthogonal to the zero mode, which can be achieved by demanding

$$\nabla^z S'_\nu(z,w)=2\pi\delta^2(z,w)-2\pi\frac{\overline{h_\nu(z)}h_\nu(w)}{\langle h_\nu\,|\,h_\nu\rangle}\ . \tag{3.203}$$

Since $h_\nu(z)$ depends holomorphically on Ω, $\tilde S_\nu$ itself will not be holomorphic in Ω. One can define an analytic propagator, at the expense of letting it transform with the wrong weight, and depend on an arbitrary point y on the Riemann surface,

$$S_\nu(z,w)=\frac{1}{E(z,w)}\frac{\sum_I\partial_I\vartheta[\nu]\left[\int_w^z\omega,\Omega\right]\omega_I(y)}{\sum_I\partial_I\vartheta[\nu](0,\Omega)\omega_I(y)}\ . \tag{3.204}$$

This propagator obeys

$$\nabla^z S_\nu(z,w)=2\pi\delta^2(z,w)\ .$$

Actually, S'_ν can be represented in terms of S,

$$S'_\nu(z,w)=S_\nu(z,w)+\frac{h_\nu(z)h_\nu(w)}{\langle h_\nu\,|\,h_\nu\rangle^2}\int d^2P\int d^2Q\,\overline{h_\nu(P)h_\nu(Q)}S_\nu(P,Q)$$
$$-\frac{h_\nu(z)}{\langle h_\nu\,|\,h_\nu\rangle}\int d^2P\,\overline{h_\nu(P)}S_\nu(P,w)+\frac{h_\nu(w)}{\langle h_\nu\,|\,h_\nu\rangle}\int d^2P\,\overline{h_\nu(P)}S_\nu(P,z)\ , \tag{3.205}$$

and does not depend on the extra point y any longer. S' is antisymmetric in z and w, as expected, and orthogonal to the zero mode. It is thus appropriate to write

$$S'_\nu(z,w)=-\langle\psi'_+(z)\psi'_+(w)\rangle\ , \tag{3.206}$$

where the prime on the fields stands for the fact that ψ_+ is considered in the space orthogonal to the zero mode.

Whereas for even-spin structure it was straightforward to show the holomorphicity of $\mathcal{C}_\nu$ in $\mathbf{z}_i$, Ω, and χ, for odd-spin structures there are several obstacles. First, the Dirac determinant with zero modes removed is no longer the absolute value square of a holomorphic function of Ω. Second, the Dirac propagator S' orthogonal to zero modes must be used to contract ψ', and it contains nonholomorphic dependences. We now show that a careful treatment actually produces a fully holomorphic amplitude $\mathcal{C}_\nu$ for odd-spin structure ν.

It is convenient to recast the contribution $\mathcal{L}_+^0+\mathcal{L}_++\mathcal{L}'_+$ in terms of a contraction over the chiral Bose field x_+, as shown in Eq. (3.191). Thus the amplitude $\mathcal{C}_\nu$ becomes

$$\mathcal{C}_\nu=Z_\Delta(\Omega)^{-10}\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi i p_I^\mu k_I^\mu\int_P^{z_i}\omega_I\right]$$
$$\times\left\langle\prod_{i=1}^n e^{ik_i\cdot x_+(z_i)}\mathcal{R}_\nu\right\rangle\ , \tag{3.207}$$

where the reduced amplitude $\mathcal{R}_\nu$ is given by

$$\mathcal{R}_\nu=\int D\psi_+e^{-I_{\psi_+}-I_m^1}e^{2\pi p_I^\mu\sigma_I^\mu}\prod_{i=1}^n e^{ik_i^\mu\theta_i\psi_+^\mu(z_i)}$$
$$=\int D\psi_+\exp\left[-I_{\psi_+}+\int d^2z\,\eta^\mu(z)\psi_+^\mu(z)\right] \tag{3.208}$$

and the source $\eta^\mu(z)$ is independent of ψ_+,

$$\eta^\mu(z)=-\frac{1}{4\pi}\chi_{\bar z}{}^+[\partial_z x_+^\mu(z)-2\pi p_I^\mu\omega_I(z)]$$
$$+i\sum_{i=1}^n k_i^\mu\theta_i\delta(z-z_i)\ . \tag{3.209}$$

We now isolate the zero mode h_ν of ψ_+,

$$\psi_+(z)=\hat{h}_\nu(z)\psi_+^0+\psi'_+(z),\quad \hat{h}_\nu(z)=\frac{h_\nu(z)}{\langle h_\nu \mid h_\nu\rangle^{1/2}} ,$$

and ψ'_+ is understood to be orthogonal to h_ν, so that the functional integral simply splits,

$$\mathcal{R}_\nu=\int d\psi_+^0 e^{\langle\eta\mid\hat{h}_\nu\rangle\psi_+^0}\int D\psi'_+ e^{-I_{\psi'_+}+\langle\eta\mid\psi'_+\rangle}$$
$$=\prod_\mu \langle\eta^\mu\mid\hat{h}_\nu\rangle(\det{}'\not{D})^5\exp\left[\tfrac{1}{2}\int\int\eta S'_\nu\eta\right] .$$

The difference between S'_ν and S_ν consists of terms proportional either to $h_\nu(z)$ or to $h_\nu(w)$. In view of the prefactor resulting from the zero-mode integration, such terms cancel. Furthermore, multilinearity of the same prefactor allows us to rearrange the normalization factor of h_ν,

$$\mathcal{R}_\nu=\prod_\mu\langle\eta^\mu\mid h_\nu\rangle\left[\frac{\det{}'\not{D}}{\langle h_\nu\mid h_\nu\rangle}\right]^5\exp\left[\tfrac{1}{2}\int\int\eta S_\nu\eta\right] . \tag{3.210}$$

As we shall see in Sec. VII.A, the determinant factor now precisely contains the correct zero-mode normalization to make it the absolute value square of a holomorphic function of Ω, and S_ν itself was of course holomorphic. Thus we have established full holomorphic splitting of the amplitudes with exponential insertions for even- and odd-spin structures.

What happens for full-fledged scattering amplitudes—say, of massless particles? There are further obstacles in principle to chiral splitting. Foremost among these is the fact that the superderivatives that enter the vertex operator construction themselves involve fields of both chiralities. This can be seen directly from Eq. (3.66), and is actually already familiar from the study of the superstring action which involves the chirality-violating term $\chi\bar{\chi}\psi_+\psi_-$. Thus the extension to higher vertex operators of the property of chiral splitting is nontrivial. In the case of massless external particles, we have checked that chiral splitting holds in exactly the same way as for simple exponential insertions, with the additional property that if ζ is the source term to $\mathcal{D}_+X$ and $\bar{\zeta}$ to $\mathcal{D}_-X$, then there will be holomorphic dependence on ζ as well. We shall not reproduce these calculations here, but postpone to the one-loop case the treatment of amplitudes of massless bosons and the proof of their chiral and holomorphic splitting properties. A general proof of these properties will be given elsewhere (D'Hoker and Phong, 1988a).

2. Spin structure versus space-time parity

It is interesting to examine the space-time character of the various amplitudes we have evaluated. Clearly, we have not directly dealt with physical external particles, but only with exponential insertions, but the observations listed below in fact easily extend to the case of any type of massless external particles, as we shall see more explicitly in the case of one loop in Sec. III.M.

From inspection of Eqs. (3.200) and (3.201), it is clear that the space-time amplitude corresponding to the *chiral half* $\mathcal{C}_\nu$ with ν even, is space-time parity conserving. External momenta and polarization tensors are contracted only with the metric tensor of space-time—the Minkowski or Euclidean metric in this case.

On the other hand, from inspection of Eqs. (3.207)–(3.210), we see that to the *chiral half* $\mathcal{C}_\nu$ with ν odd there corresponds an amplitude invariably containing a ten-dimensional space-time ϵ or completely antisymmetric tensor. It arises directly from the integration over the Dirac zero modes, which produces the product of the ten components of a Grassmann-valued space-time vector,

$$\prod_\mu\langle\eta^\mu\mid h_\nu\rangle=\frac{1}{10!}\varepsilon^{\mu_1\mu_2\cdots\mu_{10}}\eta_{\mu_1}\eta_{\mu_2}\cdots\eta_{\mu_{10}} ,$$

with $\eta_\mu=\langle\eta_\mu\mid h_\nu\rangle$. All remaining contractions of space-time indices are done with the ten-dimensional metric tensor. Thus the chiral amplitude $\mathcal{C}_\nu$ for ν odd is space-time parity violating—actually parity odd.

This means that the full amplitudes for the type-II superstring will be parity conserving if left and right worldsheet chiralities are endowed with either both even-spin or both odd-spin structure, and will be parity violating if the spin structure parities are opposite. Of course this reasoning has assumed that the vertex operators themselves do not involve the ε symbol, as is indeed always the case for low enough mass level ($m^2<12$); if it is present, the assignments should of course be reversed.

L. Tree-level amplitudes for the type-II superstring

In this section we present a reasonably complete discussion of the tree-level calculation of superstring amplitudes. To remain specific, we shall deal with the tree-level case of the type-II superstring, determine the measure, factor out the superconformal Killing vector fields, and evaluate the three and four massless boson scattering amplitudes.

For $h=0$, there are six real conformal Killing vectors, four conformal Killing spinors, and no supermoduli parameters. The measure must thus be modified to

$$DE_M^A D\Omega_M\delta(T)=(\text{sdet}'\mathcal{P}_1^\dagger\mathcal{P}_1)^{1/2}D'V^M D\Sigma DL , \tag{3.211}$$

where the prime on $D'V^M$ denotes the fact that it is restricted to the complement of the Ker$\mathcal{P}_1$. As in the bosonic case, a super Weyl transformation Σ brings out the following dependence:

$$DE_M{}^A D\Omega_M\delta(T)=(\text{sdet}'\hat{\mathcal{P}}_1^\dagger\hat{\mathcal{P}}_1)^{1/2}\frac{1}{\text{Vol}(\text{Ker}\hat{\mathcal{P}}_1)}$$
$$\times e^{-5S_{SL}(\Sigma)}D\Sigma DV^M DL . \tag{3.212}$$

Assuming that the correct procedure is to divide by the factor of $s\mathcal{N}=\text{Vol(sDiff)}\times\text{Vol(sWeyl)}\times\text{Vol(sU(1))}$, one obtains the formula for the tree-level scattering amplitudes

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle = ce^{-2\lambda}\langle\langle V_1(k_1)\cdots V_n(k_n)\rangle\rangle \times \frac{1}{\mathrm{Vol}(\mathrm{Ker}\hat{\mathcal{P}}_1)}, \tag{3.213}$$

where the symbol $\langle\langle\ \rangle\rangle$ denotes the fact that the functional integral over X alone was performed. The determinants of $\hat{\mathcal{D}}_+\hat{\mathcal{D}}_-^{(0)}$ and $\hat{\mathcal{P}}_1^+\hat{\mathcal{P}}_1$ are constants, since there are no supermoduli, and we denote their effect by c.

1. Superconformal transformations

The next issue we must settle is the volume of $\mathrm{Ker}\hat{\mathcal{P}}_1$. To analyze this, we must write down the invariant volume element on this space. The superconformal invariance group is isomorphic to complexified OSp(1,1)—the superconformal extension of PSL(2,C) defined in Eq. (2.106). To see this, we start with homogeneous coordinates ($v\ w\ \psi$), where latin (greek) variables describe (anti-commuting) commuting variables. On this triplet, we have a natural action of $GL(2\mid 1)$ T: $W\to TW$,

$$W=\begin{bmatrix} v\\ w\\ \psi\end{bmatrix},\quad T=\begin{bmatrix} a & b & \alpha\\ c & d & \beta\\ \gamma & \delta & A\end{bmatrix}. \tag{3.214}$$

To make contact with $N=1$ superspace, we introduce the projective coordinates

$$z=\frac{v}{w},\quad \theta=\frac{\psi}{w},$$

on which $GL(2\mid 1)$ acts by super Möbius transformation:

$$z\to\frac{az+b+\alpha\theta}{cz+d+\beta\theta},\quad \theta\to\frac{\gamma z+\delta+A\theta}{cz+d+\beta\theta}. \tag{3.215}$$

To obtain a *superconformal* transformation T, we must transform the line element $d\mathbf{z}=dz+\theta\, d\theta$ into itself up to a conformal scaling. Equivalently, "the quadratic form"

$$z_{12}=z_1-z_2-\theta_1\theta_2=\frac{v_1w_2-v_2w_1-\psi_1\psi_2}{w_1w_2} \tag{3.216}$$

should transform into itself up to a conformal scaling. This is uniquely achieved when the orthosymplectic form

$$K=\begin{bmatrix} 0 & +1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{3.217}$$

is left invariant under T:

$$T^TKT=K. \tag{3.218}$$

Note that the transpose of a matrix T is defined by

$$T^T=\begin{bmatrix} a & c & \gamma\\ b & d & \delta\\ -\alpha & -\beta & A\end{bmatrix},$$

so that $(TW)^T=W^TT^T$, and $\mathrm{sdet}T^T=\mathrm{sdet}T$. Thus the transformations (3.215) with T satisfying (3.218) are superconformal. The weight under which the difference transforms is easily derived, and we have

$$T: z_{12}\to\tilde{z}_{12}=\frac{z_{12}}{(cz_1+d+\beta\theta_1)(cz_2+d+\beta\theta_2)}. \tag{3.219}$$

Similarly the line element transforms as

$$d\mathbf{z}\to d\tilde{\mathbf{z}}=\frac{d\mathbf{z}}{(cz+d+\beta\theta)^2} \tag{3.220}$$

and the volume element as

$$d\mathbf{z}\wedge d\theta\to\frac{d\mathbf{z}\wedge d\theta}{cz+d+\beta\theta}. \tag{3.221}$$

Elements in OSp(1,1) are in unique correspondence with a triplet of points in the superplane $(z_1,\theta_1),(z_2,\theta_2),(z_3,\theta_3)$ obeying one single (Grassmann-valued) constraint. The counting works out because OSp(1,1) has three commuting and two anticommuting parameters. The constraint is an Osp(1,1)-invariant Grassmann-valued function, dependent on three points (Aoki, 1988), given by

$$\Delta=\frac{z_{12}\theta_3+z_{31}\theta_2+z_{23}\theta_1+\theta_1\theta_2\theta_3}{(z_{12}z_{23}z_{31})^{1/2}}. \tag{3.222}$$

The natural value for Δ is of course 0, which implies that one θ is dependent. With this value for Δ, it is easy to see that there is a unique correspondence between triplets of points satisfying $\Delta=0$ and elements of OSp(1,1), so that the latter may be accordingly parametrized.

In particular, the volume element on OSp(1,1) may be calculated in this fashion. We already know from Eqs. (3.219) and (3.220) that the six-dimensional volume element

$$\frac{dz_1dz_2dz_3d\theta_1d\theta_2d\theta_3}{(z_{12}z_{23}z_{31})^{1/2}} \tag{3.223}$$

is invariant under OSp(1,1). The invariant volume element induced on OSp(1,1) is obtained by multiplying it by the δ function of the constraint $\delta(\Delta)=\Delta$:

$$d\mu=\frac{dz_1dz_2dz_3d\theta_1d\theta_2d\theta_3}{(z_{12}z_{23}z_{31})^{1/2}}\Delta. \tag{3.224}$$

2. Evaluation of correlation functions

To calculate the correlation functions of a sequence of vertex operators, we would need the Green's function for the super Laplacian on the sphere. However, the Weyl invariance of the measure and the correlation functions, as well as the conservation of momentum, imply that one may instead work on the superplane after a stereographic projection, exactly as in the bosonic case. Here, the propagator is very simple,

$$G(\mathbf{z},\mathbf{z}')=-\ln(|z-z'-\theta\theta'|^2+\varepsilon^2), \tag{3.225}$$

and ε is understood to be infinitesimal. The vertex operators will be described extensively in Sec. VIII. Here we shall provide an example involving the simplest possible physical vertex operator: the one for bosonic particles at zero-mass level $k^2=0$,

$$V(\varepsilon,k)=g\varepsilon_{\mu;\bar\mu}\int d^2\mathbf{z}\,E\mathcal{D}_+X^\mu\mathcal{D}_-X^{\bar\mu}e^{ik\cdot X}\,, \qquad (3.226)$$

describing the graviton, the antisymmetric tensor field, and the dilaton. The polarization tensor $\varepsilon_{\mu;\bar\mu}$ is understood to be transverse in k, and the vertex is effectively normal ordered. To compute correlation functions of several of these vertices, it is useful to recall a trick known from the bosonic string. It consists of introducing a source for both X and its derivatives, and then isolating the correct expansion coefficient when developing in powers of the source. The key observation is that we may *formally* write $\varepsilon_{\mu;\bar\mu}=\zeta_\mu\bar\zeta_{\bar\mu}$, where ζ_μ and $\bar\zeta_{\bar\mu}$ are Grassmann-valued vectors. By linearity of any amplitude in the $\varepsilon_{\mu;\bar\mu}$'s, clearly any $\varepsilon_{\mu;\bar\mu}$ can be written as a formal sum, but we shall not explicitly need this construction. Once this has been done, we may introduce a generalized vertex

$$V^*(\zeta,\bar\zeta,k)=g\int d^2\mathbf{z}\,E e^{ik\cdot X+\zeta\cdot\mathcal{D}_+X+\bar\zeta\cdot\mathcal{D}_-X}\,, \qquad (3.227)$$

whose $\zeta\bar\zeta$ coefficient is precisely $V(\varepsilon,k)$. Thus we shall perform our calculations on V^*, introducing a different set of $\zeta\bar\zeta$'s for every ε of V and selecting the correct term in the expansion in ζ's.

We thus calculate the n vertex correlation function starting from the V^* operators,

$$\langle V^*(\zeta_1,\bar\zeta_1,k_1)\cdots V^*(\zeta_n,\bar\zeta_n,k_n)\rangle =g^n\int d^2\mathbf{z}_1\cdots d^2\mathbf{z}_n\left\langle \exp\int d^2\mathbf{z}\,J^\mu(\mathbf{z})X_\mu(\mathbf{z})\right\rangle\,, \qquad (3.228)$$

where the source can be read off from the definition of V^*:

$$J^\mu(\mathbf{z})=\sum_{i=1}^n(ik_i^\mu+\zeta_i^\mu\mathcal{D}_+^i+\bar\zeta_i^\mu\mathcal{D}_-^i)\delta^2(\mathbf{z},\mathbf{z}_i)\,. \qquad (3.229)$$

By completing the square in the expectation value, we get

$$\left\langle \exp\left[\int d^2\mathbf{z}\,J^\mu(\mathbf{z})X_\mu(\mathbf{z})\right]\right\rangle=(2\pi)^{10}\delta(k)\exp\left[\mathcal{G}_n-\tfrac12\sum_{i\neq j=1}^n k_i\cdot k_jG(\mathbf{z}_i,\mathbf{z}_j)\right]\,. \qquad (3.230)$$

Here the terms with $i=j$ are independent of momenta k and of the coordinates $\mathbf{z}_i$. Their contribution is absorbed into an overall normalization factor for each vertex, which will be omitted here:

$$\mathcal{G}_n=\sum_{i\neq j=1}^n(-ik_i\cdot\zeta_j\mathcal{D}_+^j-ik_i\cdot\bar\zeta_j\mathcal{D}_-^j-\tfrac12\zeta_i\cdot\zeta_j\mathcal{D}_+^i\mathcal{D}_+^j-\tfrac12\bar\zeta_i\cdot\bar\zeta_j\mathcal{D}_-^i\mathcal{D}_-^j-\tfrac12\zeta_i\cdot\bar\zeta_j\mathcal{D}_+^i\mathcal{D}_-^j-\tfrac12\bar\zeta_i\cdot\zeta_j\mathcal{D}_-^i\mathcal{D}_+^j)G(\mathbf{z}_i,\mathbf{z}_j)\,. \qquad (3.231)$$

For tree-level amplitudes, we work on the superplane and we use the Green's function of Eq. (3.225). Thus we have (with $\theta_{ij}=\theta_i-\theta_j$)

$$\begin{aligned}&\mathcal{D}_+^jG(\mathbf{z}_i,\mathbf{z}_j)=-\frac{\theta_{ij}}{z_{ij}},\quad \mathcal{D}_-^jG(\mathbf{z}_i,\mathbf{z}_j)=-\frac{\bar\theta_{ij}}{\bar z_{ij}}\,,\\ &\mathcal{D}_+^i\mathcal{D}_+^jG(\mathbf{z}_i,\mathbf{z}_j)=-\frac{1}{z_{ij}},\quad \mathcal{D}_-^i\mathcal{D}_-^jG(\mathbf{z}_i,\mathbf{z}_j)=-\frac{1}{\bar z_{ij}}\,,\\ &\mathcal{D}_+^i\mathcal{D}_-^jG(\mathbf{z}_i,\mathbf{z}_j)=0,\quad \mathcal{D}_-^i\mathcal{D}_+^jG(\mathbf{z}_i,\mathbf{z}_j)=0\,.\end{aligned} \qquad (3.232)$$

Actual calculations of the above from Eq. (3.225) would yield additional $\delta(\mathbf{z}_i,\mathbf{z}_j)$ functions, which in the tree-amplitude calculations disappear in view of analyticity in the external momenta.[23] Thus we are effectively left with

$$\mathcal{G}_n=\sum_{i\neq j=1}^n\left[+ik_i\cdot\zeta_j\frac{\theta_{ij}}{z_{ij}}+ik_i\cdot\bar\zeta_j\frac{\bar\theta_{ij}}{\bar z_{ij}}+\tfrac12\zeta_i\cdot\zeta_j\frac{1}{z_{ij}}+\tfrac12\bar\zeta_i\cdot\bar\zeta_j\frac{1}{\bar z_{ij}}\right]\,. \qquad (3.233)$$

We now work out the three-point amplitude first and separate $\mathcal{G}_n$ as a function of ζ's and $\bar\zeta$ ($\mathcal{G}_n=\mathcal{G}_n^\zeta+\mathcal{G}_n^{\bar\zeta}$):

$$\mathcal{G}_3^\zeta=i\left[+k_1\cdot\zeta_2\frac{\theta_{12}}{z_{12}}+k_2\cdot\zeta_1\frac{\theta_{12}}{z_{12}}+k_1\cdot\zeta_3\frac{\theta_{13}}{z_{13}}+k_3\cdot\zeta_1\frac{\theta_{13}}{z_{13}}+k_2\cdot\zeta_3\frac{\theta_{23}}{z_{23}}+k_3\cdot\zeta_2\frac{\theta_{23}}{z_{23}}-i\zeta_1\cdot\zeta_2\frac{1}{z_{12}}-i\zeta_1\cdot\zeta_3\frac{1}{z_{13}}-i\zeta_2\cdot\zeta_3\frac{1}{z_{23}}\right]\,. \qquad (2.234)$$

In evaluating $\exp(\mathcal{G}_n^\zeta)$, one retains terms proportional to $\zeta_1\zeta_2\zeta_3$; however, the term with three θ's vanishes because

[23]This is equivalent to the old argument of the "cancelled propagator."

$\theta_{12}\theta_{23}\theta_{31}=0$. Thus one is left with

$$\exp(\mathcal{G}_3^\zeta)\sim i\left[+\frac{\zeta_1\cdot\zeta_2}{z_{12}}\left[k_1\cdot\zeta_3\frac{\theta_{13}}{z_{13}}+k_2\cdot\zeta_3\frac{\theta_{23}}{z_{23}}\right]+\frac{\zeta_2\cdot\zeta_3}{z_{23}}\left[k_2\cdot\zeta_1\frac{\theta_{12}}{z_{12}}+k_3\cdot\zeta_1\frac{\theta_{13}}{z_{13}}\right]+\frac{\zeta_1\cdot\zeta_3}{z_{13}}\left[k_1\cdot\zeta_2\frac{\theta_{12}}{z_{12}}+k_3\cdot\zeta_2\frac{\theta_{23}}{z_{23}}\right]\right]. \tag{3.235}$$

Using transversality and momentum conservation, we have $k_2\cdot\zeta_3=-k_1\cdot\zeta_3$, etc., so that

$$\exp(\mathcal{G}_3^\zeta)\sim -i\frac{1}{z_{12}z_{23}z_{31}}[\zeta_1\cdot\zeta_2\,k_1\cdot\zeta_3(z_{23}\theta_{13}+z_{31}\theta_{23})+\text{cyclic perm.}]\,. \tag{3.236}$$

Now there is a remarkable identity:

$$z_{23}\theta_{13}+z_{31}\theta_{23}=z_{23}\theta_1+z_{31}\theta_2+z_{12}\theta_3+\theta_1\theta_2\theta_3=(z_{12}z_{23}z_{31})^{1/2}\Delta\ , \tag{3.237}$$

where Δ was the OSp(1,1)-invariant function introduced in Eq. (3.222). Thus

$$\langle\langle V(\varepsilon_1,k_1)V(\varepsilon_2,k_2)V(\varepsilon_3,k_3)\rangle\rangle=4(2\pi)^{10}\delta(k)\int\frac{d^2\mathbf{z}_1d^2\mathbf{z}_2d^2\mathbf{z}_3}{|z_{12}z_{23}z_{31}|}\Delta\bar\Delta\zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\bar\zeta_1^{\bar\mu_1}\bar\zeta_2^{\bar\mu_2}\bar\zeta_3^{\bar\mu_3}K_{\mu_1\mu_2\mu_3}K_{\bar\mu_1\bar\mu_2\bar\mu_3}\ , \tag{3.238}$$

where

$$\begin{aligned}K_{\mu_1\mu_2\mu_3}&=\eta_{\mu_1\mu_2}k_{1\mu_3}+\eta_{\mu_2\mu_3}k_{2\mu_1}+\eta_{\mu_3\mu_1}k_{3\mu_2}\ ,\\ \langle V(\varepsilon_1,k_1)V(\varepsilon_2k_2)V(\varepsilon_3k_3)\rangle&=4(2\pi)^{10}\delta(k)\varepsilon_1^{\mu_1\bar\mu_1}\varepsilon_2^{\mu_2\bar\mu_2}\varepsilon_3^{\mu_3\bar\mu_3}K_{\mu_1\mu_2\mu_3}K_{\bar\mu_1\bar\mu_2\bar\mu_3}\ .\end{aligned} \tag{3.239}$$

The factors Δ and $\bar\Delta$ appeared rather magically in the course of the above calculation. Actually, one never needs to isolate Δ or $\bar\Delta$ explicitly, provided one makes the following choice for the gauge fixing of the superconformal group:

$$z_1=0,\quad z_2=1,\quad z_3=\infty,\quad \theta_1,\quad \theta_2=0,\quad \theta_3=0\ .$$

The variable Δ in this gauge takes on the expression $\Delta=-\theta_1$, so that fixing the superconformal gauge is performed upon removal of

$$\frac{d^2z_1d^2z_2d^2\theta_2d^2z_3d^2\theta_3}{|z_{12}z_{23}z_{31}|}\ ,$$

the factor of Δ being taken care of automatically by the θ_1 integration.

To compute the four-point amplitude, we shall make use of the above gauge from the outset. We choose $z=z_1$, $z_2=0$, $z_3=1$, $z_4=\infty$, $\theta_1,\theta_2,\theta_3=\theta_4=0$, and then have

$$\begin{aligned}\mathcal{G}_4^\zeta=i\Bigg[&+k_1\cdot\zeta_2\frac{\theta_{12}}{z_{12}}+k_2\cdot\zeta_1\frac{\theta_{21}}{z_{21}}+k_1\cdot\zeta_3\frac{\theta_1}{z_{13}}+k_3\cdot\zeta_1\frac{\theta_1}{z_{13}}+k_1\cdot\zeta_4\frac{\theta_1}{z_{14}}+k_4\cdot\zeta_1\frac{\theta_1}{z_{14}}+k_2\cdot\zeta_3\frac{\theta_2}{z_{23}}+k_3\cdot\zeta_2\frac{\theta_2}{z_{23}}\\ &+k_2\cdot\zeta_4\frac{\theta_2}{z_{24}}+k_4\cdot\zeta_2\frac{\theta_2}{z_{24}}+i\zeta_1\cdot\zeta_2\frac{1}{z_{12}}+i\zeta_1\cdot\zeta_3\frac{1}{z_{13}}+i\zeta_1\cdot\zeta_4\frac{1}{z_{14}}+i\zeta_2\cdot\zeta_3\frac{1}{z_{23}}+i\zeta_2\cdot\zeta_4\frac{1}{z_{24}}+i\zeta_3\cdot\zeta_4\frac{1}{z_{34}}\Bigg]\ .\end{aligned} \tag{3.240}$$

It is easy to see that $\exp(\mathcal{G}_4^\zeta)$ contains no terms with $4k$'s because there are only two θ's. Thus

$$\begin{aligned}\exp(\mathcal{G}_4^\zeta)=&\left[\zeta_1\cdot\zeta_2\zeta_3\cdot\zeta_4\frac{1}{z_{12}z_{34}}+\zeta_1\cdot\zeta_3\zeta_2\cdot\zeta_4\frac{1}{z_{13}z_{24}}+\zeta_1\cdot\zeta_4\zeta_2\cdot\zeta_3\frac{1}{z_{14}z_{23}}\right]\\ &+\left[\zeta_1\cdot\zeta_2\left[k_1\cdot\zeta_3k_2\cdot\zeta_4\frac{\theta_2\theta_1}{z_{12}z_{13}z_{24}}+k_1\cdot\zeta_4k_2\cdot\zeta_3\frac{\theta_2\theta_1}{z_{12}z_{14}z_{23}}\right]+\text{perm.}\right]\ .\end{aligned} \tag{3.241}$$

In principle, one should now multiply this whole expression by the one involving the $\bar\zeta$'s, perform the integrals over z and θ, and regroup terms, clearly a feudal task. The calculation is enormously simplified by the factorization properties of the Veneziano integrals.

Recall that we have the ordinary integrals

$$\int\frac{d^2z}{\pi}z^A\bar z^{\tilde A}(1-z)^B(1-\bar z)^{\tilde B}=\frac{\Gamma(-1-\tilde A-\tilde B)}{\Gamma(-\tilde A)\Gamma(-\tilde B)}\frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+2)} \tag{3.242}$$

provided $A-\tilde A$ and $B-\tilde B$ are integers, which is always the case in string theory. Using the reciprocity formula for Γ

functions,

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}\ , \tag{3.243}$$

and the fact that $A-\tilde{A}$ and $B-\tilde{B}$ are integers, we see that this expression is actually symmetric under $(A,B)\leftrightarrow(\tilde{A},\tilde{B})$, as one might expect from complex conjugation. More importantly, the answer factorizes into a product of factors, each dependent only on the parameters for either the z or $\bar{z}$ coordinates. This product property implies that one need only consider, say, the z coordinates to find the full amplitude, which by the same token will also completely factorize as a function of ζ's and $\bar{\zeta}$'s. An analogous formula is derived for the superintegrals we need:

$$\int\frac{d^2\mathbf{z}_1}{\pi}d^2\theta_2[\theta_1\theta_2]^a[\bar{\theta}_1\bar{\theta}_2]^{\tilde{a}}z_{12}^{A}\bar{z}_{12}^{\tilde{A}}(1-z_1)^B(1-\bar{z}_1)^{\tilde{B}}=(-2i)^{1-a}(+2i)^{1-\tilde{a}}\frac{\Gamma(-\tilde{a}-\tilde{A}-\tilde{B})}{\Gamma(-\tilde{A})\Gamma(-\tilde{B})}\frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+1+a)} \tag{3.244}$$

Here a and $\tilde{a}$ are either 0 or 1, and the integrals are symmetric under $(aAB)\leftrightarrow(\tilde{a}\tilde{A}\tilde{B})$ using Eq. (3.243) and the fact that $A-B$ and $\tilde{A}-\tilde{B}$ are integers. With the help of Eq. (3.244), it is now straightforward to evaluate the four-point function,

$$\begin{aligned}\langle V(\varepsilon_1,k_1)V(\varepsilon_2,k_2)V(\varepsilon_3,k_3)V(\varepsilon_4,k_4)\rangle&=(2\pi)^{10}\delta(k)g^4\int d^2\mathbf{z}_1d^2\theta_2\,|z_{12}|^{-s}\,|z_1-1|^{-u}e^{\mathcal{G}_4^{\zeta}+\mathcal{G}_4^{\bar{\zeta}}}\\&=\pi(2\pi)^{10}\delta(k)g^4\frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma(1+s/2)\Gamma(1+t/2)\Gamma(1+u/2)}\varepsilon^{1\bar{1}}\varepsilon^{2\bar{2}}\varepsilon^{3\bar{3}}\varepsilon^{4\bar{4}}K_{1234}K_{\bar{1}\bar{2}\bar{3}\bar{4}}\ .\end{aligned} \tag{3.245}$$

Using the abbreviation i for μ_i to save some writing we have $K_{\mu_1\mu_2\mu_3\mu_4}=K_{1234}$, and K is then given by

$$\begin{aligned}K_{1234}=&\,(st\eta_{13}\eta_{24}-su\eta_{14}\eta_{23}-tu\eta_{12}\eta_{34})-s(k_1^4k_3^2\eta_{24}+k_2^3k_4^1\eta_{13}-k_1^3k_4^2\eta_{23}-k_2^4k_3^1\eta_{14})\\&+t(k_2^1k_4^3\eta_{13}+k_3^4k_1^2\eta_{24}-k_2^4k_1^3\eta_{34}-k_3^1k_4^2\eta_{12})-u(k_1^2k_4^3\eta_{23}+k_3^4k_2^1\eta_{14}-k_1^4k_2^3\eta_{34}-k_3^2k_4^1\eta_{12})\end{aligned} \tag{3.246}$$

We conclude this subsection by remarking that by superconformal invariance, the zero-, one-, and two-point functions of the superstring all vanish. The fastest way of obtaining this result is by remarking that SL$(2,C)$ is a subgroup of the superconformal group, and that the respective subgroups leaving 0, 1, or 2 points fixed all have infinite volume, so that the amplitudes vanish.

M. One-loop amplitudes for the type-II superstring

To deal with one-loop amplitudes, it is convenient to return to the component formulation of Sec. III.K. On the torus, there are four spin structures, one odd corresponding to periodic×periodic boundary conditions for all worldsheet spinors, and three even-spin structures, containing at least one antiperiodic boundary condition. For even-spin structure, there is one complex modulus and one complex conformal Killing vector. For odd-spin structure, there is in addition an odd modulus and a complex conformal Killing spinor. It will be convenient to represent a spin structure by its corresponding characteristics $\nu=(a,b)$. Here a and b take the value 0 or 1 according to whether the boundary conditions are antiperiodic or periodic, respectively, about A and B cycles. Left and right chiralities will be endowed with separate spin structures ν and $\bar{\nu}$. Thus it is appropriate to decompose the one-loop amplitude as follows:

$$\langle V_1\cdots V_n\rangle=\sum_{\nu\bar{\nu}}C_{\nu\bar{\nu}}\langle V_1\cdots V_n\rangle_{\nu\bar{\nu}}\ . \tag{3.247}$$

The presence of conformal Killing vectors and spinors requires the insertion of the ghost c and the superghost $\delta(\gamma_0)$ where γ_0 is the zero mode (for odd-spin structure). Thus

$$\langle V_1\cdots V_n\rangle_{\nu\bar{\nu}}=\int_{s\mathcal{M}_1}dm_K\int D(x\psi bc\beta\gamma)\mathcal{J}_\nu\bar{\mathcal{J}}_{\bar{\nu}}V_1\cdots V_ne^{-I}\ . \tag{3.248}$$

When $\nu=(1,1)$ is odd, we have

$$\mathcal{J}_\nu=bc\,\delta(\beta_0)\delta(\gamma_0)\ , \tag{3.249}$$

where β_0 and γ_0 are the zero modes of the corresponding fields. If ν is even, on the other hand, the β_0 and γ_0 modes are absent and we have

$$\mathcal{J}_\nu=bc \tag{3.250}$$

and $\bar{\mathcal{J}}_{\bar{\nu}}$ is the complex conjugate of $\mathcal{J}_\nu$, considered for spin structure $\bar{\nu}$.

We shall now evaluate this expression for the case of bosonic vertex operators. In this case, the vertex operators are independent of the ghosts, and this integral may be performed separately. Both ghost chiralities may be integrated over independently, and one recovers the formulas derived earlier. For even-spin structure ν

$$\mathcal{A}_{\rm sgh}=\int_\nu D(bc\beta\gamma)bce^{-I_{\rm sgh}}=\left|\frac{1}{A}{\det}'\nabla^z_{-1}\right|(\det\nabla^z_{-1/2})_\nu^{-1}\ , \tag{3.251}$$

whereas for odd-spin structure

$$\mathcal{A}_{\rm sgh}=\int_{(1,1)} D(bc\beta\gamma)bc\,\delta(\beta_0)\delta(\gamma_0)e^{-I_{\rm sgh}}$$
$$=\det{}'\nabla^z_{-1}(\det{}'\nabla^z_{-1/2})^{-1}_{(1,1)}=1 \ . \qquad (3.252)$$

Here A is a normalization factor for conformal Killing vectors and spinors, and is given by the area of the worldsheet: $A=2\tau_2$. Unity results in Eq. (3.252) because the operators ∇^z_{-1} and $\nabla^z_{-1/2}$ are identical on the torus (with Euclidean metric) when both have periodic boundary conditions.

It is straightforward to evaluate [for these and the matter determinants (3.257) below, we refer the reader to Sec. V.A]

$$\frac{1}{A}\det{}'\nabla^z_{-1}=\tfrac{1}{2}\eta(\tau)^2 \ ,$$

and for even-spin structure ν

$$(\det\nabla^z_{-1/2})_\nu=\frac{\vartheta[\nu](0,\tau)}{\eta(\tau)} \ . \qquad (3.253)$$

Notice that the superghost part of the amplitude is independent of the supermodulus χ. Recall indeed that the Faddeev-Popov operator could be separated into P_1 and $P_{1/2}$ without cross terms (see Sec. III.E).

1. Exponential insertions

Next, we evaluate the matter contribution, and again use the results of Sec. III.K. Recall that in principle all vertex insertions for bosonic external particles could be obtained from the insertion of (unintegrated) exponential factors. Thus it is best to evaluate these first, since they are simplest. Consider the amplitude

$$\mathcal{A}_m=\int D(x\psi)\prod_{i=1}^n e^{ik_i^\mu X^\mu(z_i,\theta_i)}e^{-I_m} \ , \qquad (3.254)$$

where $X^\mu=x^\mu+\theta\psi^\mu_+ +\bar\theta\psi^\mu_- + i\theta\bar\theta F^\mu$ and I_m is the matter action in components. It is implicit that left- and right-spin structures are fixed to be ν and $\bar\nu$. Using the results of Eqs. (3.196) and (3.197), we have

$$\mathcal{A}_m=(2\pi)^{10}\delta(k)\left[\frac{4\pi^2\det{}'\Delta}{(\mathrm{Im}\tau)^2}\right]^{-5}\left[\frac{\det{}'\mathcal{D}_+}{\langle h\mid h\rangle}\right]^5_\nu \times\left[\frac{\det{}'\mathcal{D}_-}{\langle h\mid h\rangle}\right]^5_{\bar\nu} \times\int_{\mathfrak{F}} dp^\mu \mathcal{F}'_\nu(z_i,\theta_i,\tau;p^\mu)\overline{\mathcal{F}'_{\bar\nu}(z_i,\theta_i,\tau;p^\mu)} \ , \qquad (3.255)$$

where the reduced chiral amplitude $\mathcal{F}'_\nu$ is given by

$$\mathcal{F}'_\nu=\prod_{i<j}E(z_i,z_j)^{k_i\cdot k_j}\exp(i\pi p^\mu\tau p^\mu+i2\pi p^\mu k_i^\mu z_i) \times\left\langle e^{\mathcal{L}_+ +2\pi p^\mu\sigma^\mu}\prod_{i=1}^n\exp[ik_i^\mu\theta_i\psi^\mu_+(z_i)]\right\rangle \ . \qquad (3.256)$$

The contribution of $\mathcal{L}'_+$, present in Eq. (3.196), vanishes for the torus, since there is only one χ. Further, when there are Dirac zero modes (for odd-spin structure), the last expectation value involves an integral over all of them. Finally, the determinants of the Dirac operators are understood to be primed with the zero mode h factored out when the spin structure is odd, and to have no such modification when the spin structure is even.

We have the explicit formulas

$$\frac{\det{}'\Delta}{(\mathrm{Im}\tau)^2}=|\eta(\tau)|^4 \ ,$$
$$(\det\mathcal{D}_+)_\nu=\frac{\vartheta[\nu](0,\tau)}{\eta(\tau)}, \quad \nu\neq(1,1) \ , \qquad (3.257)$$
$$\left[\frac{\det{}'\mathcal{D}_+}{\langle h\mid h\rangle}\right]_{(1,1)}=\eta(\tau)^2 \ .$$

Considerable simplification occurs upon putting the matter $\mathcal{A}_m$ and ghost $\mathcal{A}_{\rm sgh}$ parts together to obtain the full amplitude $\mathcal{A}=\mathcal{A}_m\times\mathcal{A}_{\rm sgh}\times\overline{\mathcal{A}_{\rm sgh}}$:

$$\mathcal{A}=\int D(x\psi bc\beta\gamma)\prod_{i=1}^n e^{ik_i^\mu X^\mu(z_i,\theta_i)}\mathcal{J}_\nu\bar{\mathcal{J}}_{\bar\nu}e^{-I}$$
$$=(2\pi)^{10}\delta(k)M_\nu\bar M_{\bar\nu}\int_{\mathfrak{F}}dp^\mu\mathcal{F}_\nu\bar{\mathcal{F}}_{\bar\nu} \ . \qquad (3.258)$$

For even-spin structure, we have

$$M_\nu=\frac{\vartheta[\nu](0,\tau)^4}{\eta(\tau)^{12}} \ , \qquad (3.259a)$$

whereas for odd-spin structure

$$M_{(1,1)}=1 \ . \qquad (3.259b)$$

It remains to evaluate $\mathcal{F}_\nu$. Here again, we distinguish between even- and odd-spin structure.

For even-spin structure, $\mathcal{L}_+$ and σ^μ vanish, and

$$\left\langle\prod_{i=1}^n e^{ik_i^\mu\theta_i\psi_+(z_i)}\right\rangle_{\psi_+}=\exp\left[\tfrac{1}{2}\sum_{ij}k_i\cdot k_j\theta_i\theta_j S_\nu(z_i,z_j)\right] \ , \qquad (3.260)$$

where $S_\nu(z,w)$ is the Dirac propagator, given by the Szegö kernel

$$S_\nu(z,w)=\frac{\vartheta[\nu](z-w,\tau)\vartheta_1'(0,\tau)}{\vartheta[\nu](0,\tau)\vartheta_1(z-w,\tau)} \ . \qquad (3.261)$$

It may also be useful to recall that the prime form E takes on a simple form for the torus,

$$E(z,w)=\frac{\vartheta_1(z-w,\tau)}{\vartheta_1'(0,\tau)} \ . \qquad (3.262)$$

Clearly, it is advantageous to define the "chiral X propagator" in analogy with Eq. (3.190),

$$G_\nu(\mathbf{z},\mathbf{w})=G_1(z,w)+\theta_z\theta_w S_\nu(z,w) \ ,$$
$$G_1(z,w)=-\ln E(z,w) \ , \qquad (3.263)$$

so that

$$\mathcal{F}_\nu = \exp\left[ip^\mu \tau p^\mu + i2\pi p^\mu \sum_i k_i^\mu z_i - \sum_{i<j} k_i\cdot k_j G_\nu(\mathbf{z}_i,\mathbf{z}_j)\right] . \tag{3.264}$$

Note, however, that $E(z,w)$ is multivalued around B cycles on the surfaces. The full propagator for the X field is simply related to G_ν:

$$G_{\nu\bar\nu}(\mathbf{z},\mathbf{w}) = \langle X(\mathbf{z})X(\mathbf{w})\rangle = G_\nu(\mathbf{z},\mathbf{w}) + \overline{G_{\bar\nu}(\mathbf{z},\mathbf{w})} - \frac{\pi}{2\tau_2}(z-w-\bar z+\bar w)^2 , \tag{3.265}$$

and it is well defined on the surface, though no longer meromorphic. The last term arises because of the x zero mode. No analogous terms arise for the Dirac propagator because for the even-spin structure there are no Dirac zero modes. Notice also that since the auxiliary field F^μ in X^μ has been set to zero from the outset, we do not pick up a δ-function contribution to the propagator. Analyticity in the external momenta justifies dropping such terms, as long as the propagator is evaluated between vertex operators, as will always be the case here.

For odd-spin structure, $\mathcal{L}_+$ and σ^μ do contribute; however, since they are linear in χ, each of them can only be contracted with the exponential insertion. The ψ_+ propagator $\tilde S_\nu$, orthogonal to the constant-zero mode of $\mathcal{D}$, is given by

$$\tilde S_0(z,w) = S_0(z,w) - \frac{\pi}{\tau_2}(z-w-\bar z+\bar w) ,$$
$$S_0(z,w) = \frac{\vartheta_1'(z-w,\tau)}{\vartheta_1(z-w,\tau)} . \tag{3.266}$$

Here we have abbreviated the odd-spin structure by $0=(1,1)$. It is easy to see that this is a well-defined function on the torus. The full propagator for odd-spin structure is then given by

$$G_{(1,1)}(\mathbf{z},\mathbf{w}) = \langle X(\mathbf{z})X(\mathbf{w})\rangle = G_0(\mathbf{z},\mathbf{w}) + \overline{G_0(\mathbf{z},\mathbf{w})} - \frac{\pi}{2\tau_2}(z-w-\bar z+\bar w-\theta_z\theta_w+\bar\theta_z\bar\theta_w)^2 ,$$

where

$$G_0(\mathbf{z},\mathbf{w}) = G_1(z,w) + \theta_z\theta_w S_0(z,w) .$$

2. Modular invariance

We now discuss the coefficients $C_{\nu\bar\nu}$ occurring in the summation over spin structures. $G_{\nu\bar\nu}$ is manifestly modular covariant, as may be seen by using Eq. (E5): the only effect of a modular transformation on $G_{\nu\bar\nu}$ is to permute the spin structure according to the modular group,

$$G_{\nu\bar\nu}(z-z',\theta\theta';\tau+1) = G_{\nu_1\bar\nu_1}(z-z',\theta\theta';\tau) ,$$
$$G_{\nu\bar\nu}\left[\frac{z-z'}{\tau},\frac{\theta\theta'}{\tau},-\frac{1}{\tau}\right] = G_{\nu_\tau\bar\nu_\tau}(z-z',\theta\theta';\tau) , \tag{3.267}$$

where

$$\nu_1=(a,b+a+1),\quad \bar\nu_1=(\bar a,\bar b+\bar a+1) ,$$
$$\nu_\tau=(b,a),\quad \bar\nu_\tau=(\bar b,\bar a) \pmod 2 .$$

Note that the odd-spin structure is transformed into itself. This at once implies that the vertex operator contractions $\langle\langle V_1(k_1)\cdots V_n(k_n)\rangle\rangle_{\nu\bar\nu}$ are also modular invariant in this sense. Modular invariance of the full amplitude $\langle V_1(k_1)\cdots V_n(k_n)\rangle$ will be achieved provided a choice for $C_{\nu\bar\nu}$ is made that is consistent with modular invariance. It is easily checked that the measure in Eq. (3.258) transforms correctly under modular transformations, except perhaps for a constant phase:

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_{\nu\bar\nu}(\tau+1) = (-1)^{a+\bar a}\langle V_1(k_1)\cdots V_n(k_n)\rangle_{\nu_1\bar\nu_1}(\tau) ,$$
$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_{\nu\bar\nu}\left[-\frac{1}{\tau}\right] = \langle V_1(k_1)\cdots v_n(k_n)\rangle_{\nu_\tau\bar\nu_\tau}(\tau) . \tag{3.268}$$

Hence modular invariance of the full amplitude requires the following choice for the constants $C_{\nu\bar\nu}$:

$$\tau\to-\frac{1}{\tau},\quad C_{(1,0)\bar\nu}=C_{(0,1)\bar\nu},\quad C_{\nu(1,0)}=C_{\nu(0,1)} ,$$
$$\tau\to\tau+1,\quad C_{(0,1)\bar\nu}=-C_{(0,0)\bar\nu},\quad C_{\nu(0,1)}=-C_{\nu(0,0)} , \tag{3.269}$$

and this should hold for all ν and $\bar\nu$. Note that, since the odd-spin structure (1,1) transforms as a singlet under the modular group, the relative magnitude with even-spin structures is not fixed by modular invariance. It should be determined by factorization, in the limit where the torus degenerates to the sphere.

3. Three- and four-point amplitudes for massless bosons

Though the prescriptions given above are complete and explicit, it may be instructive to work things out for an example. Let us consider scattering amplitudes with massless external particles only (the graviton, dilaton, and antisymmetric tensor field). Such operators are produced by the generating vertex $V^*(\zeta,\bar\zeta;k)$ introduced in Eq. (3.227). As in the case of tree level, the amplitude (3.228) is expressed through Eqs. (3.230) and (3.231), but the propagator is now understood to be $G_{\nu\bar\nu}$, of Eq. (3.265).

For even-spin structure, we consider the chirality-conserving form first and then split it to obtain the chiral amplitude. The relevant superderivatives are

$$\mathcal{D}_+^j G_{\nu\bar\nu}(\mathbf{z}_i,\mathbf{z}_j) = \mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j) + \frac{\pi}{\tau_2}\theta_j(z_i-z_j-\bar z_i+\bar z_j) ,$$
$$\mathcal{D}_+^i\mathcal{D}_+^j G_{\nu\bar\nu}(\mathbf{z}_i,\mathbf{z}_j) = \mathcal{D}_+^i\mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j) + \frac{\pi}{\tau_2}\theta_i\theta_j , \tag{3.270}$$
$$\mathcal{D}_+^i\mathcal{D}_-^j G_{\nu\bar\nu}(\mathbf{z}_i,\mathbf{z}_j) = -\frac{\pi}{\tau_2}\theta_i\bar\theta_j .$$

Again, we have neglected all $\delta(z_i,z_j)$'s, because they do not contribute to the amplitude in view of analyticity in the external momenta. Thus we may separate $\mathcal{G}_n$ of Eq. (3.231) into two chiral parts expressed only in terms of the chiral propagator G_ν (and its complex conjugate) and a mixed part, which we shall call $\hat{\mathcal{G}}_n$:

$$\mathcal{G}_n-\tfrac{1}{2}\sum_{i\neq j}k_i\cdot k_j G_{\nu\bar\nu}(\mathbf{z}_i,\mathbf{z}_j) = \mathcal{G}_n^{\zeta}+\mathcal{G}_n^{\bar\zeta}+\hat{\mathcal{G}}_n -\tfrac{1}{2}\sum_{i\neq j}k_i\cdot k_j[G_\nu(\mathbf{z}_i,\mathbf{z}_j)+\overline{G_{\bar\nu}(\mathbf{z}_i,\mathbf{z}_j)}]\ , \qquad (3.271)$$

where the chiral part is given by

$$\mathcal{G}_n^{\zeta}=\sum_{i\neq j}[-ik_i\cdot\zeta_j\mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j)-\tfrac{1}{2}\zeta_i\cdot\zeta_j\mathcal{D}_+^i\mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j)] \qquad (3.272)$$

and $\mathcal{G}_n^{\bar\zeta}$ is its complex conjugate (for imaginary k_i^μ). The mixed part can be simplified with the use of rearrangements familiar from Sec. III.K:

$$\hat{\mathcal{G}}_n=-\frac{2\pi}{\tau_2}\left[\sum_i[-\mathrm{Im}(\zeta_i^\mu\theta_i)+ik_i^\mu\mathrm{Im}z_i]\right]^2 .$$

Following the derivation of Sec. III.K, we may introduce the loop momenta p^μ and write

$$e^{\hat{\mathcal{G}}_n}=(\tau_2)^5\int_{\mathfrak{F}}dp^\mu\,|\exp[\,i\pi p^\mu\tau p^\mu +2\pi p^\mu(-\zeta_i^\mu\theta_i+ik_i^\mu z_i)]\,|^2 . \qquad (3.273)$$

Thus the full amplitude (still for even-spin structure) may be recast in a familiar form,

$$\langle V_1^*\cdots V_n^*\rangle_{\nu\bar\nu}=(2\pi)^{10}\delta(k)M_\nu\overline{M}_{\bar\nu}\int_{\mathfrak{F}}dp^\mu\mathcal{F}_\nu\overline{\mathcal{F}}_{\bar\nu}\ , \qquad (3.274)$$

where

$$\mathcal{F}_\nu(z_i,k,\zeta,\tau;p^\mu) =\exp\left[\,i\pi p^2\tau+2\pi p^\mu(-\zeta_i^\mu\theta_i+ik_i^\mu z_i) -\tfrac{1}{2}\sum_{i\neq j}k_i\cdot k_j G_\nu(z_i,z_j)+\mathcal{G}_n^{\zeta}\right] . \qquad (3.275)$$

Of course this amplitude should now be integrated over moduli space.

To evaluate the zero-, one-, two-, three-, four-, and five-point amplitudes, the above is in fact enough, for only the even-spin structures contribute to their amplitudes. Indeed, for the odd-spin structure the Dirac operator has one (chiral) zero mode for each dimension of space-time $d=10$; there is thus a total of ten zero modes. Inserting, for example, a massless vertex eats up two zero modes. However, one fermion mode is also eaten up by fixing a conformal spinor gauge for the supersymmetry operator. One more is produced by the presence of the supermoduli parameters. All zero modes must of course be killed, so naively the lowest number of vertex operator insertions necessary to make the amplitude nonzero is five. However, overall momentum conservation implies that this amplitude also vanishes, and one has to go to six external particles to obtain a nonzero contribution from the odd-spin structure.

We first show that the zero-, one-, two-, and three-point functions vanish identically. This fact is based upon two fundamental observations. For three or fewer external massless particles, one always has $k_i\cdot k_j=0$ for all i and j, so that $\mathcal{F}_\nu$ only involves $\mathcal{G}_n^{\zeta}$, which depends on the derivatives of G_ν only. These derivatives are given by

$$\mathcal{D}_+^i G_\nu(\mathbf{z}_i,\mathbf{z}_j;\tau)=\theta_j S_\nu(z_i,z_j)+\theta_i\partial_i G_1(z_i,z_j)\ ,$$
$$\mathcal{D}_+^i\mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j;\tau)=-S_\nu(z_i,z_j)+\theta_i\theta_j\partial_i\partial_j G_1(z_i,z_j)\ . \qquad (3.276)$$

The partition function and the one- and two-particle amplitudes all vanish simply by the use of the famous Jacobi identity of (E11) and the assignments of the coefficients $C_{\nu\bar\nu}$:

$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4=0,\quad \nu=(a,b),\quad \bar\nu=(\bar a,\bar b)\ . \qquad (3.277)$$

For the three-point function one uses, in addition to the above, the facts that

$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4\mathcal{D}_+^1G_\nu(1,2)\mathcal{D}_+^2G_\nu(2,3)\mathcal{D}_+^3G_\nu(3,1)=0\ ,$$
$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4\mathcal{D}_+^1G_\nu(1,2)\mathcal{D}_+^2G_\nu(2,1)\mathcal{D}_+^3G_\nu(3,1)=0\ , \qquad (3.278)$$

which are equivalent—in component language—to the equations

$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4S_\nu(z_1-z_2)S_\nu(z_2-z_3)S_\nu(z_3-z_1)=0\ ,$$
$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4S_\nu(z_1-z_2)S_\nu(z_2-z_1)=0\ . \qquad (3.279)$$

All these identities are easily proven with the help of Eqs. (3.277) and (E7').

The calculation of the four-point function is more involved, and ϑ-function identities are heavily used. There are three types of terms: those with four factors of k, those with two factors of k, and those without explicit k's at all contracted onto the polarization tensors. Our first task is to show that the terms with four factors of k cancel after summation over all spin structures. One needs the following Riemann-type identity:

$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4\mathcal{D}_+^1G_\nu(1,i_1)\mathcal{D}_+^2G_\nu(2,i_2)\mathcal{D}_+^3G_\nu(3,i_3)\mathcal{D}_+^4G_\nu(4,i_4) =\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4S_\nu(1,i_1)S_\nu(2,i_2)S_\nu(3,i_3)S_\nu(4,i_4)\ . \qquad (3.280)$$

To establish this, use the representation (3.276) for the derivatives: terms with four G_1's cancel because of (3.277), terms with three G_1's due to transversality, and terms with two or one due to (3.279). Permutations $(1,2,3,4)\to(i_1,i_2,i_3,i_4)$ which leave one or more points fixed need not be considered, as their contribution cancels due to transversality of the polarization tensors. The remaining nine permutations cancel in view of the Riemann identity (E7').

Now we calculate the terms with two momenta k; it is useful to take an example. Consider terms arising as the coefficient of

$$\zeta_1\cdot\zeta_2\zeta_3\cdot k_{i_3}\zeta_4\cdot k_{i_4}\ ,$$

where i_3 and i_4 are different from three and four, respectively. The spin structure sum is then again simplified, with the help of the Riemann identities, and one finds

$$\sum_\nu C_{\nu\bar\nu}\vartheta_\nu^4(0,\tau)\mathcal{D}_+^1\mathcal{D}_+^2G_\nu(1,2)\mathcal{D}_+^3G_\nu(3,i_3)\mathcal{D}_+^4G_\nu(4,i_4)\prod_{i<j}[1+k_i\cdot k_j\theta_i\theta_jS_\nu(i,j)]$$
$$=\theta_{i_3}\theta_{i_4}\sum_\nu C_{\nu\bar\nu}\vartheta_\nu(0,\tau)^4S_\nu(1,2)S_\nu(3,i_3)S_\nu(4,i_4)\prod_{i<j}(\ )\ , \qquad (3.281)$$

where the last factor arises from the expansion of the superspace Green's function. Since we must end up with four θ's, the product $\prod_{i<j}$ produces only terms with two θ's, so that the answer will be a linear function of s, t, and u. With some further use of the Riemann identities, one can evaluate it rather easily, and one finds

$$\zeta_1\cdot\zeta_2(t\zeta_3\cdot k_1\zeta_4\cdot k_2+u\zeta_3\cdot k_2\zeta_4\cdot k_1)\ . \qquad (3.282)$$

Upon inspection, one notices that this result is reminiscent of the tree-level answer obtained in Eq. (3.246). One can now easily complete the analysis by checking that the other terms also have the same form as the tree-level answer. Thus our final expression for the one-loop four-point function in the type-II superstring is

$$\langle V(\varepsilon_1,k_1)\cdots V(\varepsilon_4,k_4)\rangle=g^4\delta(k)\mathcal{A}_1\varepsilon^{1\bar1}\varepsilon^{2\bar2}\varepsilon^{3\bar3}\varepsilon^{4\bar4}$$
$$\times K_{1234}K_{\bar1\bar2\bar3\bar4}\ , \qquad (3.283)$$

where the reduced amplitude is given by

$$\mathcal{A}_1=\int_{\mathcal{M}_1}\frac{d^2\tau}{2\tau_2^2}\frac{1}{(\tau_2)^4}\int d^2z_1d^2z_2d^2z_3d^2z_4$$
$$\times|F_{12}F_{34}|^{-s/2}|F_{23}F_{14}|^{-t/2}$$
$$\times|F_{13}F_{24}|^{-u/2}\ . \qquad (3.284)$$

We have used identity (E9) and we have abbreviated $F_{ij}=F(z_i,z_j)$, where the function F was defined in Eq. (2.91) or (3.183). Overall translation invariance on the torus allows us to integrate over one of the four positions, so that we may set $z_4=0$ and

$$\mathcal{A}_1=\frac{1}{2}\int_{\mathcal{M}_1}\frac{d^2\tau}{(\tau_2)^5}\int d^2z_1d^2z_2d^2z_3\left|\frac{F_{12}F_{34}}{F_{13}F_{24}}\right|^{-s/2}$$
$$\times\left|\frac{F_{23}F_{14}}{F_{13}F_{24}}\right|^{-t/2}\ , \qquad (3.285)$$

which agrees with the classic formula derived in the operator formalism.

Several remarks are in order here. First, it is remarkable that the kinematical form for the one-loop amplitude coincides with that for the tree-level amplitude. Second, our calculation of the one-loop four-point amplitude is perhaps more involved than when it is performed in the light-cone operator formalism. However, it has to be recalled that the corresponding calculation in the light-cone formulation was simple only for graphs with very few external legs, ultimately becoming unwieldy for graphs with more than six legs. In our covariant RNS formulation, the difficulty increases, but only slightly so.

4. Higher-point amplitudes and odd-spin structure

Let us now come back to the case of odd-spin structure and derive explicit formulas for scattering amplitudes of massless particles. There are three additional complications as compared to the even-spin structure case. First, we have an odd modulus to integrate over (constant χ), second there is a (constant) Dirac zero mode, third since there is a Dirac zero mode, the chiral amplitude analogous to $\mathcal{F}$ (but now with massless vertex insertions) is no longer holomorphic in τ and χ, but there are mixed terms. We shall tackle these issues by evaluating the matter contribution of the path integral with generating functions for massless operators inserted at points z_i,θ_i, which we do not integrate over.

We begin with the nonchiral amplitude for odd-spin structure

$$\mathcal{A}=\mathcal{A}_{\text{sgh}}\int D(x\psi)\prod_{i=1}^n\exp(ik_i^\mu X^\mu+\zeta_i^\mu\mathcal{D}_+X^\mu$$
$$+\bar\zeta_i^\mu\mathcal{D}_-X^\mu)e^{-I_m}\ . \qquad (3.286)$$

Recall that the superghost contribution was unity for odd-spin structure: $\mathcal{A}_{\text{sgh}}=1$.

Care has to be taken to include the full superderivatives in this expression, since the χ field does not vanish now. To be specific, if $X^\mu=x^\mu+\theta\psi_+^\mu+\bar\theta\psi_-^\mu+i\theta\bar\theta F^\mu$, we get

$$\mathcal{D}_+X=\psi_++i\bar\theta F+\theta(\partial_zx+\tfrac12\chi_z{}^-\psi_-)$$
$$+\theta\bar\theta(-\tfrac14\chi_{\bar z}{}^+\chi_z{}^-\psi_++\tfrac12\chi_z{}^-\partial_{\bar z}x+D_z\psi_+)$$
$$(3.287)$$

and $\mathcal{D}_- X$ is its complex conjugate.

Integration over the x field is performed as before, and we find a complicated expression due to the presence of several contributions from the vertex. However, there is a remarkable partial cancellation with the ψ_+- and ψ_--dependent terms in the vertex, which considerably simplifies the final answer. Some further partial contractions of fermionic insertions ultimately lead to

$$\mathcal{A}=(2\pi)^{10}\delta(k)\left[\frac{4\pi^2\det'\Delta}{(\tau_2)^2}\right]^{-5} e^{\mathcal{L}_+^0+\mathcal{L}_-^0+\mathcal{L}_+^1+\mathcal{L}_-^1}(\mathrm{Im}\tau)^{-5} \times\int D\psi_+ D\psi_- e^{\mathcal{L}_+^2+\mathcal{L}_-^2} \times\exp\left[-\frac{2\pi}{\tau_2}(\mathrm{Im}\sigma^\mu+ik_i^\mu \mathrm{Im}z_i)^2\right] . \quad (3.288)$$

We have also used the following abbreviations:

$$\mathcal{L}_+^0=\sum_{i<j} k_i\cdot k_j \ln E(z_i,z_j) ,$$

$$\mathcal{L}_+^1=\sum_{i<j}[\theta_i\theta_j\zeta_i\cdot\zeta_j\partial_{z_i}\partial_{z_j}\ln E(z_i,z_j) -2ik_i\cdot\zeta_j\theta_j\partial_{z_j}\ln E(z_i,z_j)] , \quad (3.289)$$

$$\mathcal{L}_+^2=\sum_i\left[ik_i^\mu\theta_i\psi_+^\mu(z_i) -\frac{1}{4\pi}\zeta_i^\mu\theta_i\int d^2w\,\chi_{\bar z}{}^+\psi_+^\mu(w)\partial_{z_i}\partial_w\ln E(z_i,w) +\zeta_i^\mu\psi_+^\mu(z_i) -\frac{1}{4\pi}k_i^\mu\int d^2w\,\chi_{\bar z}{}^+\psi_+^\mu(w)\partial_w\ln E(z_i,w)\right] .$$

Note that $\mathcal{L}_+^0$ and $\mathcal{L}_+^1$ are independent of the fermion field, whereas $\mathcal{L}_+^2$ is chiral in the sense used throughout. We have also defined

$$\sigma^\mu=\frac{1}{4\pi}\int d^2z\,\chi_{\bar z}{}^+\psi_+^\mu(z)-\sum_i\zeta_i^\mu\theta_i . \quad (3.290)$$

Since $\chi_{\bar z}{}^+$ is a constant, only the zero mode ψ_+^0 of ψ_+ contributes to σ^μ. Notice that the amplitude $\mathcal{A}$ is chirally split in ψ_+ and ψ_-, except for its zero modes. Thus it is necessary to isolate these zero modes explicitly, which is achieved by splitting $\psi_\pm=\psi'_\pm+\psi_\pm^0$. The contractions of the nonzero modes must then be performed with the propagator $\tilde S_0$ of Eq. (3.266), which is indeed orthogonal to constants. One readily finds that

$$\mathcal{A}=(2\pi)^{10}\delta(k)(\mathrm{Im}\tau)^{-5}e^{\mathcal{L}_+^0+\mathcal{L}_-^0+\mathcal{L}_+^1+\mathcal{L}_-^1+\mathcal{L}_+^3+\mathcal{L}_-^3} \times\int d\psi_+^{0\mu}d\psi_-^{0\mu}e^{Z}\prod_{i=1}^n e^{v_i^\mu\psi_+^{0\mu}+\bar v_i^\mu\psi_-^{0\mu}} , \quad (3.291)$$

where we use the abbreviation $v_i^\mu=ik_i^\mu\theta_i+\zeta_i^\mu\theta_i$,

$$Z=-\frac{2\pi}{\tau_2}\sum_i[-\mathrm{Im}(\zeta_i^\mu\theta_i)+ik_i^\mu\mathrm{Im}z_i +(v_i^\mu\chi_{\bar z}{}^+ +\bar v_i^\mu\chi_z{}^-)\mathrm{Im}z_i]^2 . \quad (3.292)$$

Contraction of the ψ_+-dependent terms $\mathcal{L}_+^2$ produces also a chiral part $\mathcal{L}_+^3=\frac{1}{2}\langle\mathcal{L}_+^2\mathcal{L}_+^2\rangle$, where all the ψ contractions have been carried out with the propagator S_0 instead of $\tilde S_0$. This function is explicitly given by

$$\mathcal{L}_+^3=\sum_{ij}\left[-\tfrac{1}{2}v_i^\mu v_j^\mu S_0(z_i,z_j)+\frac{1}{4\pi}v_i^\mu\zeta_j^\mu\theta_j\chi_{\bar z}{}^+\int d^2w\,\partial_{z_j}\partial_w\ln E(z_i,w)S_0(z_i,w) +\frac{1}{4\pi}v_i^\mu k_j^\mu\chi_{\bar z}{}^+\int d^2w\,\partial_w\ln E(z_j,w)S_0(z_i,w)\right] . \quad (3.293)$$

The nonmanifestly chiral terms arising in the full contraction have been lumped into Z. Now we see that the amplitude again splits when we introduce the internal momenta p^μ. Putting all these together, we find

$$\mathcal{A}=(2\pi)^{10}\delta(k)\int_{\mathfrak{F}}dp^\mu\int d\psi_+^{0\mu}d\psi_-^{0\mu}\mathcal{F}_0\bar{\mathcal{F}}_0 , \quad (3.294)$$

where the reduced chiral amplitude $\mathcal{F}_0$ is given by

$$\mathcal{F}_0=\exp\left[\mathcal{L}_+^0+\mathcal{L}_+^1+\mathcal{L}_+^3+i\pi\tau p^2+2\pi p^\mu\zeta_p^\mu +\psi_+^{0\mu}\sum_i v_i^\mu\right] , \quad (3.295)$$

where

$$\zeta_p^\mu=\sum_i(-\zeta_i^\mu\theta_i+ik_i^\mu z_i-\chi_{\bar z}{}^+ v_i^\mu z_i) . \quad (3.296)$$

Note that the presence of the zero-mode integral ensures that $\sum_i v_i^\mu$ vanishes at all intermediate steps in the derivation of those formulas. Furthermore, it guarantees that ζ_p^μ be invariant under overall translations in z_i, as it should.

This answer looks rather complicated, but in fact the combination of $\mathcal{L}_+^0$, $\mathcal{L}_+^1$, and $\mathcal{L}_+^3$ can be obtained from a very simple recipe. Start with the functional integral (3.286), but instead of using full superderivatives $\mathcal{D}_+$ and $\mathcal{D}_-$, rather use the flat superderivatives ∂_+ and ∂_- alone, and use the propagators G_0 and S_0 instead of the full propagators. Also ignore all possible complications that could arise because of zero modes to the various fields. Thus we can symbolically write

$$\exp(\mathcal{L}_+^0+\mathcal{L}_+^1+\mathcal{L}_+^3)=\left\langle e^{-I_m^1}\prod_{i=1}^n e^{ik_i^\mu X_+^\mu+\zeta_i^\mu \partial_+ X_+^\mu}\right\rangle , \tag{3.297}$$

where all the fields and propagators are now "chiral"

$$\begin{aligned} X_+^\mu &= x_+^\mu+\theta\psi_+^\mu ,\\ \partial_+ X_+^\mu &= \psi_+^\mu+\theta\partial_z x_+^\mu ,\\ \langle x_+(z)x_+(w)\rangle &= -\ln E(z,w)=G_1(z,w) ,\\ \langle \psi_+(z)\psi_+(w)\rangle &= \partial_z \ln E(z,w)=S_0(z,w) . \end{aligned} \tag{3.298}$$

In fact, one may also introduce a full chiral superfield propagator, including the effects of the supermodulus

$$\begin{aligned} G_0(\mathbf{z},\mathbf{w}) &= \langle X_+(\mathbf{z})X_+(\mathbf{w})\rangle_{\text{full}}\\ &= -\ln\frac{\vartheta_1(z-w-\theta_z\theta_w,\tilde\tau)}{\vartheta_1'(0,\tilde\tau)} , \end{aligned} \tag{3.299}$$

where $\tilde\tau=\tau-\chi_{\bar z}{}^+(\theta_z+\theta_w)$. The amplitude is then given by

$$\begin{aligned} \mathcal{L}_+^0+\mathcal{L}_+^1+\mathcal{L}_+^3 &= -\tfrac{1}{2}(ik_i^\mu+\zeta_i^\mu\partial_+^i)(ik_j^\mu+\zeta_j^\mu\partial_+^j)\\ &\quad\times G_0(\mathbf{z}_i,\mathbf{z}_j) . \end{aligned} \tag{3.300}$$

One-loop amplitudes for four-graviton scattering have been computed in the operator formalism by Green and Schwarz (1982) and Schwarz (1982) for the type-II string. Space-time supersymmetry breaking to one-loop order was investigated by Rohm (1984). For the heterotic string, one-loop four-point functions were calculated by Gross *et al.* (1986) and Yashikozawa (1986, 1987) for gauge bosons, Sakai and Tanii (1987) for gravitons, and Cai and Nunez (1987) for gravitons, gauge bosons, and antisymmetric tensor fields. The first two works rely on the operator method, the third on path integrals. Our present method based on path integrals is more complicated than the operator method for a small number of external states (up to six), but it remains tractable as that number increases.

Issues of modular invariance are addressed by Witten (1984), Arnaudon *et al.* (1987), Gliozzi (1987), and Parkes (1987). Generating functions for anomalies as modular forms are introduced in Schellekens and Warner (1986, 1987), Pilch, Schellekens, and Warner (1987), and Witten (1987). Nonrenormalization theorems were stated in Martinec (1986) and shown explicitly to apply in the one-loop case by Tanii (1985, 1986), and Namazie, Narain, and Sarmidi (1986). The hexagon anomaly was shown to vanish to one loop in the heterotic string for gauge groups Spin(32)/Z_2 and $E_8\times E_8$ by Gross and Mende (1987a). The "supertheta" function of Eq. (3.299) also occurs in Freund and Rabin (1988).

Open-string amplitudes to one loop are discussed in the report of Schwarz (1982) and more recently in Frampton, Moxhay, and Ng (1985), Clavelli (1986), Frampton, Kikuchi, and Ng (1986), Burgess (1987), and Kostelecky, Lechtenfeld, and Samuel (1987).

N. Heterotic strings

The heterotic string was constructed by Gross, Harvey, Martinec, and Rohm (1985a, 1985b, 1986) as a hybrid of one chiral half of the type-II string (say left chirality) and one half of the closed bosonic string, compactified on a 16-dimensional torus T^{16}. As a string theory, it lives in ten space-time dimensions, and we may alternatively regard it as a theory of ten bosonic degrees of freedom x^μ, ten Majorana-Weyl worldsheet spinors ψ_+^μ (left chirality), and a number of fields representing the internal degrees of freedom. These could be 16 bosonic (right-chirality) x^a or, when fermionized, 32 right-chirality Majorana-Weyl spinors ψ_-^a. It is in terms of the latter that we had written the heterotic string worldsheet action of Eq. (3.3). We shall repeat it here for convenience:

$$\begin{aligned} I_m &= I_H+I_i ,\\ I_H &= \frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(D_z x^\mu D_{\bar z}x^\mu-\psi_+^\mu D_{\bar z}\psi_+^\mu\\ &\qquad +\chi_{\bar z}{}^+\psi_+^\mu D_z x^\mu) , \end{aligned} \tag{3.301}$$

where I_i is the action for the internal degrees of freedom,

$$I_i=\frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(-\psi_-^a D_z\psi_-^a),\quad a=1,\ldots,32 \tag{3.302a}$$

when written in fermionic representation, and

$$I_i=\frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,D_z x^a D_{\bar z}x^a,\quad a=1,\ldots,16 \tag{3.302b}$$

when written in bosonic representation, and it is understood that only the chiral halves of the bosonic contributions are kept. This action exhibits $N=\frac{1}{2}$ local supersymmetry invariance and may be quantized as a supergravity theory in its own right. In Sec. III.N.1 we shall give a brief account of this approach, without entering into any details. Instead we shall rather study the heterotic string as a cross breeding of half a type-II string and half a (partially) compactified bosonic string. An advantage of the latter approach is that we can gain direct information about the torus T^{16} or, equivalently, about the lattice[24] Λ out of which the torus is constructed: $T^{16}=R^{16}/\Lambda$. This second approach will be discussed more extensively in Sec. III.N.2. Incidentally, it has already been stressed, when discussing super Weyl anomalies, that in a worldsheet chirality-nonconserving theory, super Weyl invariance must cancel for both left and right chiralities separately. This is equivalent to cancellation of super Weyl and local U(1) anomalies of the whole theory. Clearly, this requires that 16 internal bosons x^a or 32 internal Majorana-Weyl fermions ψ_-^a be present, as discussed before, when the critical dimension is $d=10$. The structure of the lattice Λ is at this point

[24]We shall always assume that Λ is indeed 16 dimensional.

left open, and will be narrowed down—through insistence on modular invariance—to the root lattice of $E_8 \times E_8$ or of Spin(32)/Z_2.

1. $N=\frac{1}{2}$ supergeometry

$N=\frac{1}{2}$ superspace is parametrized by two commuting coordinates ξ and $\bar{\xi}$ and one anticommuting θ, collected into a supercoordinate $z^M=(\xi,\bar{\xi},\theta)$. The U(1) frame is similarly reduced to $A=(z,\bar{z},+)$. Covariant derivatives, torsion, and curvature are defined as in Eqs. (3.7) and (3.10), but the torsion constraints (3.11) are now restricted to the $A=(z,\bar{z},+)$. Using the Bianchi identities, one then has

$$T_{++}{}^{+}=T_{\bar{z}+}{}^{A}=T_{z+}{}^{+}=T_{++}{}^{\bar{z}}=0,\quad T_{++}{}^{z}=2\ ,$$
$$T_{z\bar{z}}{}^{+}=-\frac{i}{2}R_{\bar{z}+},\quad R_{z\bar{z}}=-\mathcal{D}_+R_{\bar{z}+},\quad R_{++}=R_{z+}=0\ , \qquad (3.303)$$

so that all components of the torsion and curvature are expressible in terms of $R_{\bar{z}+}$. The transformation laws of these fields under super-reparametrizations, super Weyl transformations, and local U(1) transformations may be readily obtained by restriction of the $N=1$ case, and we shall not rewrite them here.

A difficult feature of the $N=\frac{1}{2}$ supergeometry is that the supercurvature field $R_{\bar{z}+}$ now has U(1) weight-$\frac{1}{2}$ and is anticommuting, so that there is no sense to setting it to a constant other than zero. In view of the super Gauss-Bonnet formula analogous to that for $N=1$ supergeometry, $R_{\bar{z}+}$ should not vanish whenever $\chi(M)\neq 0$. Asking $R_{\bar{z}+}$ to be covariantly constant now leads to nontrivial differential equations. Thus it is not clear in the case of heterotic geometry how the geometric ideas discussed in the case of $N=1$ supergeometry can be implemented; as a matter of fact, it is not clear that they can be.

The superspace action for the heterotic string is

$$I=\frac{1}{4\pi}\int d^2\xi\, d\theta(\mathrm{sdet}E_M{}^{A})(\mathcal{D}_+X^\mu\mathcal{D}_{\bar{z}}X^\mu+\Psi^a\mathcal{D}_+\Psi^a)\ , \qquad (3.304)$$

where X^μ is the even superfield $X^\mu(\xi,\bar{\xi};\theta)=x^\mu+\theta\psi_+^\mu$ and Ψ^a is the odd superfield $\Psi^a=\psi_-^a+\bar{\theta}F^a$, with ψ_+^μ the space-time fermions, ψ_-^a the internal fermions, and F^a an auxiliary field.

$N=\frac{1}{2}$ supergeometry was investigated by Hull and Witten (1985), Brooks, Muhammad and Gates (1986), Gates, Brooks, and Muhammad (1987), Nelson and Moore (1986), and Evans and Ovrut (1986a, 1986b, 1987).

2. Heterosis

The fundamental idea behind heterosis is that the left- and right-moving degrees of freedom on the worldsheet are described by independent degrees of freedom, sharing only their common overall momentum. The notion of left- and right-movers may be understood on a compact surface with a metric of Euclidean signature as analytic and antianalytic, or for fermionic degrees of freedom of course as left and right chirality. Unfortunately, the notions of left- and right-movers or analytic and antianalytic are defined only when the fields satisfy their equations of motion. They do not *a priori* make sense in a functional integral formulation where all fields are to be integrated over. This is especially a problem for the bosonic fields x^μ or x^a which are real.

In our discussion of the type-II string, we have already had to separate left- and right-chirality components in order to endow them with separate spin structures. We have actually achieved much more. When loop momenta p_I^μ are fixed, and for a fixed point in supermoduli space, the integrand splits as a function that is analytic in the period matrix Ω_{IJ}, analytic in the positions of the vertex insertions z_i, and dependent only on $\chi_{\bar{z}}{}^{+}$, times its complex conjugate.[25] This chiral splitting at fixed internal momenta will be reconsidered in much more detail in Sec. VII and identified there with holomorphic splitting at fixed internal momenta on supermoduli space. The holomorphic structure of supermoduli space is that introduced in Sec. III.G, and it will be shown in Sec. VII that Ω_{IJ} and $\chi_{\bar{z}}{}^{+}$ are holomorphic coordinates for supermoduli. This holomorphic splitting points to a way of identifying the contributions of the right-movers in the bosonic x^μ. In fact, the closed bosonic string amplitudes could be split in a similar fashion, even though x^μ is not a chiral field. Again, at fixed internal momenta, the integrand is the absolute-value square of a function analytic in Ω_{IJ} and in the positions of the vertex operator insertions z_i. (Of course we will have to check that this kind of splitting continues to hold when the closed bosonic string is compactified on a torus T^{16}.) The right-movers' contributions can now be taken to be the antiholomorphic factor. The vertex operators (at fixed positions) for the heterotic string are similarly constructed of half a type-II vertex and half a bosonic vertex. Actually, this is not quite so, because each contains pieces of both chirality. However, in the end, all pieces can be put together and split when the internal momenta on the string are kept fixed.

Thus the recipe for heterosis will be to take the left chiral half of the type-II string and the right chiral half of the bosonic string at the *same internal momenta* and to multiply them together and integrate over the internal momenta.

That this prescription is the correct one is confirmed by the fact that it alone will reproduce quantized $N=\frac{1}{2}$

[25] Recall that the complex conjugate is in general evaluated for a different spin structure. Also recall that all momenta—internal p_I^μ and external k_i^μ—have been analytically continued to imaginary values.

supergravity from the chirally split type-II superstring. Indeed, the amplitudes for the heterotic geometry may be gotten by setting $\chi_{\bar{z}}{}^{+}=0$, so that we can read off from Eq. (3.196) that an exponential insertion would give

$$\mathcal{A}_H=(2\pi)^{10}\delta(k)\int_{\mathfrak{J}} dp_I^\mu \mathcal{F}_\nu(\mathbf{z}_i,\psi_+,\Omega,\chi;p_I^\mu)\times\overline{\mathcal{B}_{10}(z_i,\Omega;p_I^\mu)}\,\overline{\mathcal{B}_{16}(z_i,\Omega)}\;, \qquad (3.305)$$

where $\mathcal{F}_\nu$ was defined in Eq. (3.197) and the ten-dimensional chiral bosonic amplitude is given by

$$\mathcal{B}_{10}(z_i,\Omega;p_I^\mu)=Z_\Delta(\Omega)^{-10}\prod_{i<j}E(z_i,z_j)^{k_i\cdot k_j}\times\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi i p_I^\mu k_i^\mu\int_P^{z_i}\omega_I\right]. \qquad (3.306)$$

The symbols are the same as in the case of the type-II string analysis of Sec. III.K.

We shall now derive an expression for the contribution to the amplitude $\mathcal{B}_{16}$ of the internal degrees of freedom. We begin with the fermionic representation, described by the action (3.302a). For convenience, we shall consider its complex conjugate, so as to obtain $\mathcal{B}_{16}$ directly. We shall also restrict ourselves to considering only insertions of ψ^a and not its derivatives, which is enough for the case of vertex operators for massless particles. Furthermore, all 32 fermions ψ^a are decoupled from one another, so we shall evaluate the contributions of a single one first, endowed with spin structure $\bar{\nu}$. Actually, the 32 fermions were understood to be Majorana-Weyl, which is not realizable on a worldsheet with Euclidean signature. Thus we shall pair them two by two and endow these with the same spin structure. We then have

$$\mathcal{B}_{\bar{\nu}}^1=\int_{\bar{\nu}}D\psi\prod_{i=1}^n e^{\eta_i\psi(z_i)}e^{-I_\psi}\;. \qquad (3.307)$$

For even-spin structure, this integral has no zero modes, and we get

$$\mathcal{B}_{\bar{\nu}}^1=(\det \not{D}_+)_{\bar{\nu}}\exp\left[+\tfrac{1}{2}\sum_{ij}\eta_i\eta_j S_{\bar{\nu}}(z_i,z_j)\right], \qquad (3.308)$$

where $S_{\bar{\nu}}$ is the Dirac propagator already encountered in Eq. (3.202) and given by the Szegö kernel. The Dirac determinant will be evaluated using bosonization methods in Sec. VII, and we just quote here the answer from Eq. (7.61):

$$(\det \not{D}_+)_{\bar{\nu}}=Z_\Delta(\Omega)^{-1}\vartheta[\bar{\nu}](0,\Omega)\;, \qquad (3.309)$$

very much in analogy with the one-loop formula of Eq. (3.253).

For odd-spin structure, there is generically one zero-mode $h_{\bar{\nu}}$, and the chiral Dirac propagator is given by Eq. (3.206). Thus

$$\mathcal{B}_\nu^1=\int d\psi^0\int D\psi' e^{-I_{\psi'}}\exp\left[\sum_i\eta_i\psi^0\right]\prod_{i=1}^n e^{\eta_i\psi'(z_i)}=\left[\sum_i\eta_i h_{\bar{\nu}}(z_i)\right](\det{}'\not{D}_+)_{\bar{\nu}}\times\exp\left[+\tfrac{1}{2}\sum_{ij}\eta_i\eta_j\tilde{S}_{\bar{\nu}}(z_i,z_j)\right], \qquad (3.310)$$

where $(\det{}'\not{D}_+)_{\bar{\nu}}$ is the chiral half of $(\det{}'\not{D}/\langle h_\nu\,|\,h_\nu\rangle)_{\bar{\nu}}$. Due to the overall factor linear in η, $\tilde{S}_{\bar{\nu}}$ in the exponential is equivalent to $S_{\bar{\nu}}(z,w)$ of Eq. (3.204) in view of Eq. (3.205), so that Eq. (3.310) is analytic in z_i, η_i, and Ω, but also well defined on the surface.

Now that we have evaluated the contribution of a single (complex) fermion, it remains to put the 16 copies together. This must be done in a modular-invariant fashion. Recall that in the type-II string one had to sum independently over the spin structures assigned to left and right chirality. Each chirality sector was responsible for a space-time supersymmetry, all by itself, so that the theory exhibits $N=2$ supersymmetry. In the heterotic string, left and right chiralities are very different objects, and one could sum separately over the spin structures of left and right chirality, where right chirality now encompasses the internal degrees of freedom. One might also imagine linking the spin structure sum for left and right chirality. In the latter case, it should be expected that space-time supersymmetry would be destroyed. This leaves open a vast class of possibilities, which is narrowed down by the requirement of modular invariance and spin statistics. Seiberg and Witten (1986) have argued that modular invariance requires the fermions ψ^a to have the same spin structure in groups of eight (or four of our complexified ones). This eight is familiar from the modular transformation properties of the ϑ function, which always involves an eighth root of unity. This indeed occurs when the ψ's all carry a space-time index. However, in that case, they describe both bosonic and fermionic space-time degrees of freedom. Since internal ψ^a's should describe only space-time bosonic degrees of freedom, the ψ^a's should actually have the same spin structure in groups of 16 (or eight of our complexified ones). Hence the internal degrees of freedom must exhibit a symmetry that contains SO(16)×SO(16).

When the spin structure of left and right chirality are intertwined in a nontrivial fashion, one will in fact obtain an SO(16)×SO(16) string that is modular invariant (at least to one loop) but not supersymmetric. This type of string theory was investigated by Dixon and Harvey (1986), Seiberg and Witten (1986), and Alvarez-Gaumé *et al.* (1986). Its compactifications were explored by Ginsparg and Vafa (1987).

On the other hand, if spin structures for left and right chirality are summed over independently, then $N=1$ supersymmetry is maintained. The general expression for the internal amplitude is

$$\mathcal{B}_{16}=\sum_{\bar{\nu}_1\bar{\nu}_2}C_{\bar{\nu}_1\bar{\nu}_2}(\mathcal{B}_{\bar{\nu}_1}^1)^8(\mathcal{B}_{\bar{\nu}_2}^1)^8\;. \qquad (3.311)$$

Under a modular transformation

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

even- and odd-spin structures are mapped into themselves, so we may limit our discussion to the even case. Hence

$$\begin{aligned} M(\mathcal{B}_{16}) &= [\det(C\Omega+D)]^{-8}\sum_{\bar v_1 \bar v_2} C_{\bar v_1 \bar v_2}(\mathcal{B}^1_{M\bar v_1})^8(\mathcal{B}^1_{M\bar v_2})^8 \\ &= [\det(C\Omega+D)]^{-8}\sum_{\bar v_1 \bar v_2} C_{M^{-1}\bar v_1 M^{-1}\bar v_2}(\mathcal{B}^1_{\bar v_1})^8(\mathcal{B}^1_{\bar v_2})^8 \end{aligned} \tag{3.312}$$

and

$$C_{\bar v_1 \bar v_2}=C_{M^{-1}\bar v_1 M^{-1}\bar v_2} \tag{3.313}$$

for all M. If $\bar v_1 \neq \bar v_2$, then let M fix $\bar v_1$. This reduces the modular group from Sp$(2h,\mathbf{Z})$ to Sp$(2h-2,\mathbf{Z})$. This is enough for us to see that

$$C_{\bar v_1 \bar v_2}=C_{\bar v_1 \bar v_2'} \quad \text{if } \bar v_2 \neq \bar v_1 \neq \bar v_2' .$$

Since C is symmetric, all off-diagonal elements in C must be equal. On the other hand, taking $\bar v_1=\bar v_2$, we see that all on-diagonal elements must be equal as well. Thus there are two independent solutions. All $C_{\bar v_1 \bar v_2}$ are equal for all $\bar v_1$ and $\bar v_2$, and

$$\mathcal{B}_{16}=\left[\sum_{\bar v}(\mathcal{B}^1_{\bar v})^8\right]^2 \tag{3.314a}$$

or all off-diagonal elements of C vanish, so that

$$\mathcal{B}'_{16}=\sum_{\bar v}(\mathcal{B}^1_{\bar v})^{16} . \tag{3.314b}$$

In the latter case, we see that all ψ^a's are endowed with the same spin structure, thus exhibiting Spin(32) symmetry.

Now let us consider the one-loop partition function only, and evaluate the above partial amplitudes:

$$\sum_{\bar v}(\mathcal{B}^1_{\bar v})^8=\vartheta^8_{00}(0,\tau)+\vartheta^8_{01}(0,\tau)+\vartheta^8_{10}(0,\tau) . \tag{3.315}$$

This is a modular form of weight 4. With the help of Jacobi's theorem on the number $r_4(n)$ of representations of an integer n as a sum of four squares $r_4(n)=8\sigma_1(n)$ one easily finds that the above sum of three theta functions equals

$$1+240\sum_n \sigma_3(n)e^{i\pi n\tau}, \quad \sigma_\alpha(n)=\sum_{d\,|\,n} d^\alpha , \tag{3.316}$$

which is the theta function for the root lattice of E_8. Hence $\mathcal{B}_{16}$ is the amplitude for the group $E_8\times E_8$, and $\mathcal{B}'_{16}$ for Spin(32)/Z_2.

Next we derive an expression for the contribution of internal degrees of freedom to the same amplitude $\mathcal{B}_{16}$ in terms of the bosonic variable x^a. When we compactify the closed bosonic string on a torus T^{16}, the $x^a(z)$ field is no longer single valued, but is shifted by a lattice vector of Λ as z moves around a homology cycle,

$$x^a(\gamma z)=x^a(z)+T^a_\gamma, \quad T^a_\gamma \in \Lambda .$$

We may interpolate T^a_γ with the use of a harmonic function $T^a_\gamma(z)$ and introduce a single-valued field y^a,

$$x^a(z)=y^a(z)+T^a_\gamma(z) .$$

Hence we can represent the differential dx^a as

$$dx^a(z)=dy^a(z)+\sum_I [m^a_I h^A_I(z)+n^a_I h^B_I(z)] , \tag{3.317}$$

where h^A and h^B are harmonic (real) one-forms, normalized to

$$\oint_{A_I} h^A_J=\oint_{B_I} h^B_J=\delta_{IJ}, \quad \oint_{A_I} h^B_J=\oint_{B_I} h^A_J=0 .$$

The vectors m^a_I and n^a_I belong to Λ and determine the winding number of the Riemann surface in T^{16}. The action in a given topological sector is now easily computed, and one finds

$$I_x(x)=I_x(y)+\frac{\pi}{2}(n^a_I-m^a_K\bar\Omega_{KI})(\mathrm{Im}\,\Omega)^{-1}_{IJ}(n^a_J-\Omega_{JL}m^a_L) . \tag{3.318}$$

We are now going to make the following assumptions concerning m_I and n_I and the lattice they lie on. First, we assume that m_I and n_I run throughout the full lattice. Hence, if λ_α are the 16 basis vectors generating Λ, then

$$m^a_I=m^\alpha_I\lambda^a_\alpha, \quad n^a_I=n^\alpha_I\lambda^a_\alpha ,$$

where m^α_I and n^α_I run over all integers. We shall denote the lattice metric by $g_{\alpha\beta}=\lambda_\alpha\cdot\lambda_\beta$, and furthermore restrict ourselves to lattices for which the volume of the unit cell is one: $\det g_{\alpha\beta}=1$. Finally, we assume that the entries of $g_{\alpha\beta}$ are integers; since $\det g_{\alpha\beta}=1$, this means that $g^{\alpha\beta}$ also has integer entries. When all the above requirements are met, then the amplitude

$$\mathcal{A}=\int Dx^a \prod_{i=1}^n e^{iK^a_i x^a(z_i)}e^{-I_x} \tag{3.319}$$

will be Weyl invariant, provided the external momenta satisfy $K_i^2=2$ so that the lattice must be even. The lattice metric $g_{\alpha\beta}$ can now be viewed as the Cartan matrix of a Lie algebra, and since $g_{\alpha\beta}$ is symmetric, the possible Lie algebras are SO$(2n)$ and E^8 or products thereof. The amplitude $\mathcal{A}$ is easily worked out:[26]

$$\mathcal{A}=(2\pi)^{16}\delta(K)\int dP^a_I \sum_{a_I,b_I}\left|\mathcal{B}\begin{bmatrix}\delta'_I\\ \delta''_I\end{bmatrix}(z_i,\Omega,P^\mu_I)\right|^2 . \tag{3.320}$$

Here δ'^α_I and δ''^α_I are half-order characteristics and take values 0 and $\frac{1}{2}$. The reduced amplitude is given by

[26]This formula is a special case of toroidal compactifications considered in collaboration with V. Periwal.